2.3 Calculating Limits Using the Limit Laws

Calculating limits by testing values of $x$ close to $a$ is tedious. The following Theorem essentially says that any ‘nice’ combination of functions has exactly the limit you’d expect.

**Theorem.** Suppose $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and that $c$ is constant. Then the following limits exist and may be computed.

1. $\lim_{x \to a} c = c$
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
5. $\lim_{x \to a} (f(x)g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
6. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ (provided $\lim_{x \to a} g(x) \neq 0$)

The Theorem also holds for one-sided limits and, with a little care\(^1\) for infinite limits. For example, if $\lim_{x \to 2} f(x) = -3$ and $\lim_{x \to 2} g(x) = -\infty$, then

$$\lim_{x \to 2} [f(x) + g(x)] = -\infty \quad \text{and} \quad \lim_{x \to 2} f(x)g(x) = \infty$$

**Corollary.** Suppose that $p(x) = c_na^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$ is a polynomial, then $\lim_{x \to a} p(x) = p(a)$. Moreover, if $r$ is any rational function, and $a \in \text{dom}(r)$, then $\lim_{x \to a} r(x) = r(a)$.

**Proof.** Simply calculate:

$$\lim_{x \to a} p(x) = \lim_{x \to a} c_nx^n + \cdots + c_1x + c_0 = c_n \lim_{x \to a} x^n + \cdots + c_1 \lim_{x \to a} x + c_0 = c_na^n + \cdots + c_1a + c_0 = p(a)$$

If $r$ is rational, then $r(x) = \frac{p(x)}{q(x)}$ for some polynomials $p, q$. Rule 6 now finished things off. \(\blacksquare\)

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\(^1\)If you end up with an indeterminate form $0 \div 0$, $\infty - \infty$, etc., then the rules don’t apply. We will deal with these limits later using l’Hôpital’s Rule.
Examples

1. Suppose that \( \lim_{x \to a} f(x) = 3 \), \( \lim_{x \to a} g(x) = -1 \), \( \lim_{x \to a} h(x) = \infty \), and \( \lim_{x \to a} h(x) = 6 \). Then
   \[
   \lim_{x \to a} f(x) + 3g(x) = 3 + 3(-1) = 0
   \]
   \[
   \lim_{x \to a} 2f(x)g(x)h(x) = 2 \cdot 3 \cdot (-1) \lim_{x \to a} h(x) = -\infty
   \]
   \[
   \lim_{x \to a} 2f(x)g(x)h(x) = 2 \cdot 3 \cdot (-1) \cdot 6 = 36
   \]
   \[
   \lim_{x \to a} \frac{g(x)}{f(x)h(x)} = \frac{-1}{3 \lim_{x \to a} h(x)} = 0
   \]
   \[
   \lim_{x \to a} \frac{g(x)}{f(x)h(x)} = \frac{-1}{3 \cdot 6} = \frac{-1}{18}
   \]

2. Simple evaluation:
   \[
   \lim_{x \to 1} \frac{x^3 + 2x^2 - x - 1}{4x^2 - 1} = \frac{1 + 2 - 1 - 1}{4 - 1} = \frac{1}{3}
   \]

3. Factorizing:
   \[
   \lim_{x \to 2} \frac{x^2 - 7x + 10}{x^2 - 4x + 4} = \lim_{x \to 2} \frac{(x-2)(x-5)}{(x-2)(x-2)} = \lim_{x \to 2} \frac{x-5}{x-2} = -3 \lim_{x \to 2} \frac{1}{x-2} = \infty
   \]

Roots and Rationalizing

**Theorem.** \( \lim_{x \to a} f(x) = L \implies \lim_{x \to a} \sqrt[3]{f(x)} = \sqrt[3]{L} \)

Recall how you would convert an expression with surds in the denominator into one with surds in the numerator:

\[
\frac{4}{3 + \sqrt{5}} = \frac{4}{3 + \sqrt{5}} \cdot \frac{3 - \sqrt{5}}{3 - \sqrt{5}} = \frac{4(3 - \sqrt{5})}{9 - 5} = 3 - \sqrt{5}
\]

A similar approach can be used for limits.

Examples

1. \( \lim_{x \to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x} \) yields the indeterminate form \( \frac{0}{0} \). Multiplying by \( 1 = \frac{\sqrt{x + 3} + \sqrt{3}}{\sqrt{x + 3} + \sqrt{3}} = 1 \) fixes the problem:
   \[
   \lim_{x \to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x} = \lim_{x \to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x} \cdot \frac{\sqrt{x + 3} + \sqrt{3}}{\sqrt{x + 3} + \sqrt{3}}
   \]
   \[
   = \lim_{x \to 0} \frac{x + 3 - 3}{x(\sqrt{x + 3} + \sqrt{3})}
   \]
   \[
   = \lim_{x \to 0} \frac{1}{\sqrt{x + 3} + \sqrt{3}} = \frac{1}{2\sqrt{3}}
   \]

2. \( \lim_{x \to 4} \frac{\sqrt{x^2 + 9} - 5}{x - 4} = \frac{4}{5} \)
Comparing Limits and the Squeeze Theorem

While simple limits can be computed using the basic limit laws, more complicated functions are often best treated by comparison.

**Theorem.** Suppose that \( f(x) \leq g(x) \) for all \( x \neq a \) and suppose that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) both exist. Then
\[
\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)
\]

**Theorem (Squeeze Theorem).** Suppose that \( f(x) \leq g(x) \leq h(x) \) for all \( x \neq a \), and that \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \). Then \( \lim_{x \to a} g(x) \) exists and also equals \( L \).

**Example** In this example we compare the complicated function \( g(x) = x \sin \left( \frac{1}{x^2} \right) \) with the much simpler function \( f(x) = |x| \). Since \(-1 \leq \sin \left( \frac{1}{x^2} \right) \leq 1\) for all \( x \neq 0 \), we have that
\[
-|x| \leq x \sin \left( \frac{1}{x^2} \right) \leq |x|
\]
Since \( \lim_{x \to 0} |x| = 0 \) it follows that
\[
\lim_{x \to 0} x \sin \left( \frac{1}{x^2} \right) = 0
\]

**Homework**

1. (a) Prove that \( x - y = (x^{1/3} - y^{1/3})(x^{2/3} + x^{1/3}y^{1/3} + y^{2/3}) \).

(b) Hence or otherwise compute the limit \( \lim_{x \to 8} \frac{\sqrt{x} - 2}{x - 8} \).

2. Suppose that \( f(x) < g(x) \) for all \( x \neq a \) and that limits of \( f \) and \( g \) both exist at \( x = a \). Give an example which shows that we may only conclude that \( \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \). That is, the inequality need not be strict.

3. Show that \( \lim_{x \to 0} x^2 \cos \left( \frac{1}{x} \right) \) exists and compute it.