2.3 Calculating Limits Using the Limit Laws

Calculating limits by testing values of x close to a is tedious. The following Theorem essentially says that any 'nice' combination of functions has exactly the limit you'd expect.

Theorem. *Suppose* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *both exist and that* c *is constant. Then the following limits exist and may be computed.*

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
5. $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
6. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \big(\textit{provided } \lim_{x\to a} g(x) \neq 0\big)$

The Theorem also hold for one-sided limits and, with a little care,[1] for infinite limits. For example, if $\lim_{x\to 2} f(x) = -3$ and $\lim_{x\to 2} g(x) = -\infty$, then

$$\lim_{x\to 2}[f(x) + g(x)] = -\infty \qquad \text{and} \qquad \lim_{x\to 2} f(x)g(x) = \infty$$

Corollary. *Suppose that* $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ *is a polynomial, then* $\lim_{x\to a} p(x) = p(a)$. *Moreover, if* r *is any rational function, and* $a \in \operatorname{dom}(r)$, *then* $\lim_{x\to a} r(x) = f(a)$.

Proof. Simply calculate:

$$\begin{aligned}
\lim_{x\to a} p(x) &= \lim_{x\to a} c_n x^n + \cdots + c_1 \lim_{x\to a} x + \lim_{x\to a} c_0 && \text{(rule 4)}\\
&= c_n \lim_{x\to a} x^n + \cdots + c_1 \lim_{x\to a} x + c_0 && \text{(rule 3)}\\
&= c_n a^n + \cdots + c_1 a + c_0 && \text{(rule 5 (repeatedly) and 2)}\\
&= p(a)
\end{aligned}$$

If r is rational, then $r(x) = \dfrac{p(x)}{q(x)}$ for some polynomials p, q. Rule 6 now finished things off. ■

[1] If you end up with an *indeterminate form* $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty$, etc., then the rules don't apply. We will deal with these limits later using l'Hôpital's Rule.

Examples

1. Suppose that $\lim_{x\to a} f(x) = 3$, $\lim_{x\to a} g(x) = -1$, $\lim_{x\to a^-} h(x) = \infty$, and $\lim_{x\to a^+} h(x) = 6$. Then

$$\lim_{x\to a} f(x) + 3g(x) = 3 + 3(-1) = 0$$
$$\lim_{x\to a^-} 2f(x)g(x)h(x) = 2\cdot 3\cdot(-1)\lim_{x\to a^-} h(x) = -\infty$$
$$\lim_{x\to a^+} 2f(x)g(x)h(x) = 2\cdot 3\cdot(-1)\cdot 6 = 36$$
$$\lim_{x\to a^-} \frac{g(x)}{f(x)h(x)} = \frac{-1}{3\lim_{x\to a^-} h(x)} = 0$$
$$\lim_{x\to a^+} \frac{g(x)}{f(x)h(x)} = \frac{-1}{3\cdot 6} = -\frac{1}{18}$$

2. Simple evaluation: $\lim_{x\to 1} \frac{x^3+2x^2-x-1}{4x^2-1} = \frac{1+2-1-1}{4-1} = \frac{1}{3}$

3. Factorizing: $\lim_{x\to 2^-} \frac{x^2-7x+10}{x^2-4x+4} = \lim_{x\to 2^-} \frac{(x-2)(x-5)}{(x-2)(x-2)} = \lim_{x\to 2^-} \frac{x-5}{x-2} = -3\lim_{x\to 2^-} \frac{1}{x-2} = \infty$

Roots and Rationalizing

Theorem. $\lim_{x\to a} f(x) = L \Longrightarrow \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$

Recall how you would convert an expression with surds in the denominator into one with surds in the numerator:

$$\frac{4}{3+\sqrt{5}} = \frac{4}{3+\sqrt{5}}\cdot\frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{4(3-\sqrt{5})}{9-5} = 3-\sqrt{5}$$

A similar approach can be used for limits.

Examples

1. $\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$ yields the indeterminate form $\frac{0}{0}$. Multiplying by $1 = \frac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}} = 1$ fixes the problem:

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x} &= \lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}\cdot\frac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}} \\ &= \lim_{x\to 0} \frac{x+3-3}{x(\sqrt{x+3}+\sqrt{3})} \\ &= \lim_{x\to 0} \frac{1}{\sqrt{x+3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}\end{aligned}$$

2. $\lim_{x\to 4} \frac{\sqrt{x^2+9}-5}{x-4} = \frac{4}{5}$

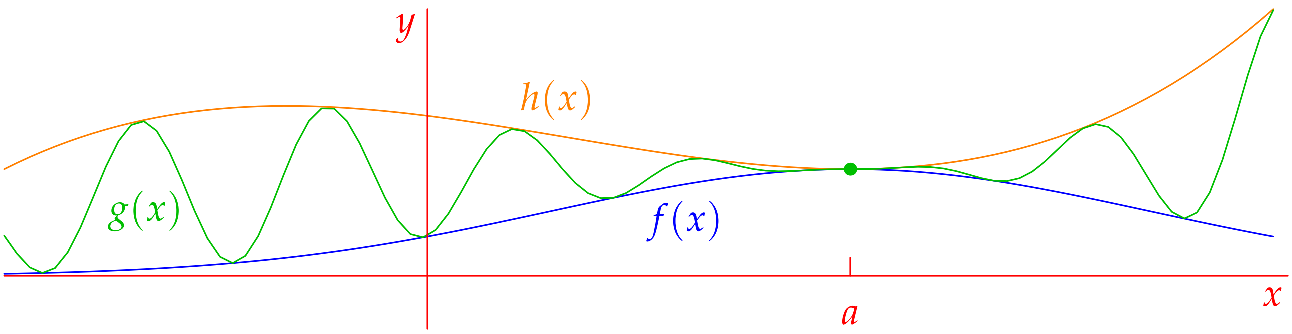
Comparing Limits and the Squeeze Theorem

While simple limits can be computed using the basic limit laws, more complicated functions are often best treated by *comparison*.

Theorem. *Suppose that $f(x) \le g(x)$ for all $x \neq a$ and suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

Theorem (Squeeze Theorem)**.** *Suppose that $f(x) \le g(x) \le h(x)$ for all $x \neq a$, and that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x)$ exists and also equals L.*

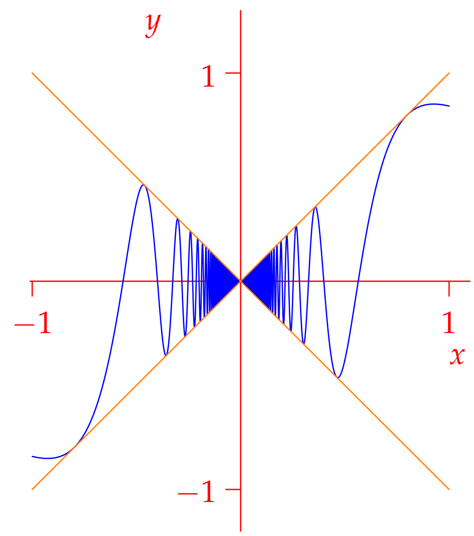


Example In this example we compare the complicated function $g(x) = x\sin\left(\frac{1}{x^2}\right)$ with the much simpler function $f(x) = |x|$. Since $-1 \le \sin\left(\frac{1}{x^2}\right) \le 1$ for all $x \neq 0$, we have that

$$-|x| \le x\sin\left(\frac{1}{x^2}\right) \le |x|$$

Since $\lim_{x\to 0} |x| = 0$ it follows that

$$\lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right) = 0$$



Homework

1. (a) Prove that $x - y = (x^{1/3} - y^{1/3})(x^{2/3} + x^{1/3}y^{1/3} + y^{2/3})$.

   (b) Hence or otherwise compute the limit $\lim_{x\to 8} \frac{\sqrt[3]{x} - 2}{x - 8}$

2. Suppose that $f(x) < g(x)$ for all $x \neq a$ and that limits of f and g both exist at $x = a$. Give an example which shows that we may only conclude that $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$. That is, the inequality need not be strict.

3. Show that $\lim_{x\to 0} x^2 \cos(\frac{1}{x})$ exists and compute it.