2.4 The Precise Definition of a Limit (Optional & Non-examinable!)

In our earlier definition of limit we did not explain what the terms ‘approaches’ or ‘arbitrarily close’ mean. The concept of arbitrarily close in mathematics works something like a game. To say that one can find a number arbitrarily close to \( L \), one must be able to give an example when told:

“Give a number closer to \( L \) than a distance \( \varepsilon \)

regardless of how small a distance \( \varepsilon \) is given.

The idea of a limit \( \lim_{x \to a} f(x) = L \) is that one can force the distance between \( f(x) \) and \( L \) to be as small as one likes by choosing the distance between \( x \) and \( a \) to be small enough.

**Definition.** Suppose \( f \) is a function defined on an interval containing \( x = a \), but not necessarily at \( a \). We say that \( f \) has limit \( L \) as \( x \) approaches \( a \) if:

For all \( \varepsilon > 0 \) there is some \( \delta > 0 \) such that

\[
0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon
\]

\[(\dagger)\]

\[\lim_{x \to a} f(x) = L\]

Regardless of the \( \varepsilon \) we are given, we can find some \( \delta \) which satisfies \((\dagger)\)

It is usually very difficult to find an explicit formula for a suitable \( \delta \) in terms of \( \varepsilon \): the Definition is instead used to prove a few basic examples and all of the limit laws and theorems from previous sections.\footnote{The details are covered in the first two weeks of an Upper Division Analysis course.}

**Example** We prove that \( \lim_{x \to 2} x^2 = 4 \).

Let \( \varepsilon > 0 \) be given, and define \( \delta = \min(\frac{\varepsilon}{3}, 1) \).

If \( 0 < |x - 2| < \delta \), then \( |x - 2| < 1 \implies x + 2 < 3 \), and so

\[
|x^2 - 4| = |(x - 2)(x + 2)| = |x - 2||x + 2| < \delta \cdot 3 \leq \varepsilon.
\]

and so \( \lim_{x \to 2} x^2 = 4 \).

How did we come up with the choice of \( \delta = \min(\frac{\varepsilon}{3}, 1) \)? Scratch-work and creativity! Indeed it is far from the only suitable choice.
No Limit In this picture the left- and right-limits are different, hence there is no limit at \( x = a \). How can we view this in terms of \( \epsilon \) and \( \delta \)?

\[
\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L' \]

\( \lim_{x \to a} f(x) = L \) says that \( L \) is the only possible candidate for the limit.

Suppose we were given the indicated value \( \epsilon \). Regardless of our choice of \( \delta > 0 \), we will be able to find values of \( x \) (in the blue region) which satisfy both

\[
0 < |x - a| < \delta \quad \text{and} \quad |f(x) - L| \geq \epsilon
\]

The definition of limit does not hold for all \( \epsilon > 0 \), and so the limit does not exist.

**Homework**

1. Suppose that \( \lim_{x \to a} f(x) = L \). That is, for all given \( \epsilon > 0 \), there is some \( \delta > 0 \) for which

\[
0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.
\]

Let \( c \neq 0 \) be constant and assume that \( \epsilon > 0 \) is given. Show that there exists \( \delta > 0 \) for which

\[
0 < |x - a| < \delta \implies |c f(x) - cL| < \epsilon.
\]

This proves that \( \lim_{x \to a} c f(x) = cL \). The other limit laws are proved similarly.