2.4 The Precise Definition of a Limit (Optional & Non-examinable!)

In our earlier definition of limit we did not explain what the terms 'approaches' or 'arbitrarily close' mean. The concept of arbitrarily close in mathematics works something like a game. To say that one can find a number arbitrarily close to L, one must be able to give an example when told:

"Give a number closer to L than a distance ε"

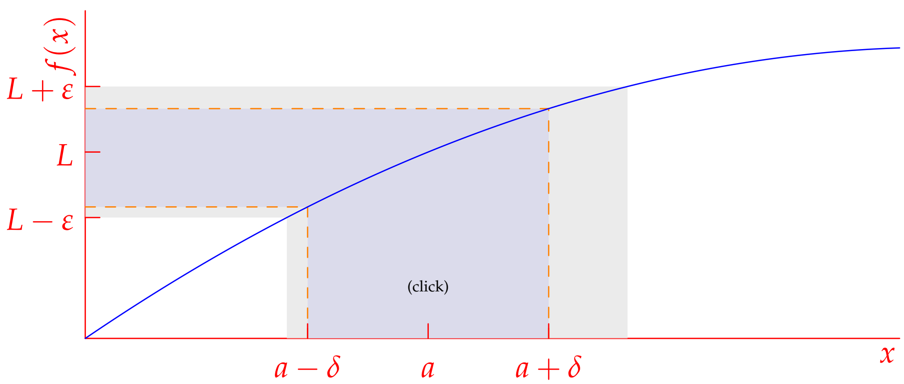
regardless of how small a distance ε is given.

The idea of a limit $\lim_{x\to a} f(x) = L$ is that one can *force* the distance between $f(x)$ and L to be as small as one likes by *choosing* the distance between x and a to be small enough.

Definition. *Suppose f is a function defined on an interval containing $x = a$, but not necessarily at a. We say that f has limit L as x approaches a if:*

For all $\varepsilon > 0$ there is some $\delta > 0$ such that

$$0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon \tag{†}$$



$\lim_{x\to a} f(x) = L$

Regardless of the ε we are given, we can find some δ which satisfies (†)

It is usually very difficult to find an explicit formula for a suitable δ in terms of ε: the Definition is instead used to *prove* a few basic examples and all of the limit laws and theorems from previous sections.[1]

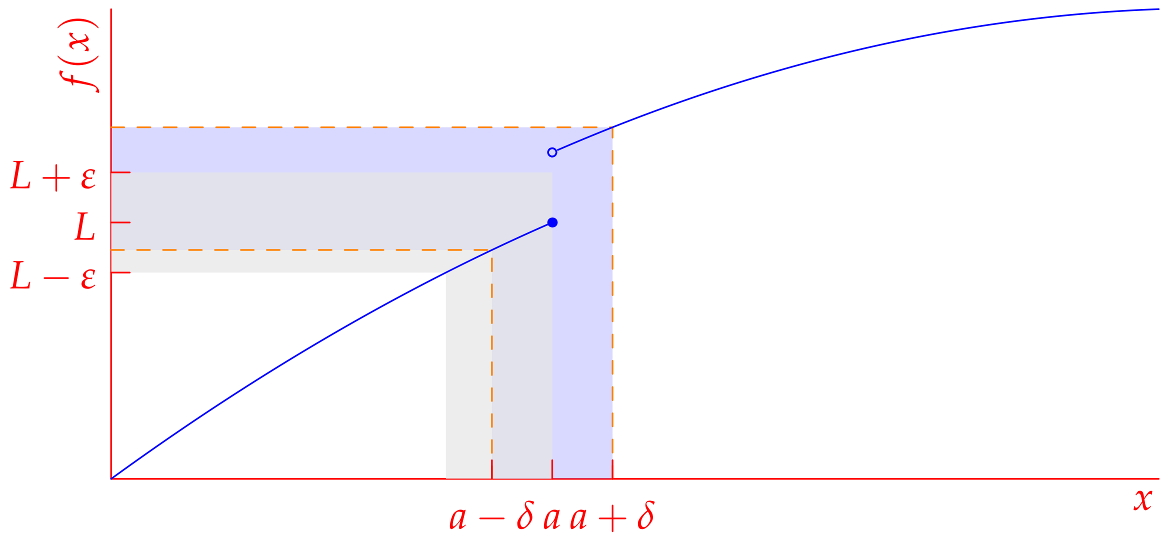
Example We *prove* that $\lim_{x\to 2} x^2 = 4$.
Let $\varepsilon > 0$ be given, and define $\delta = \min(\frac{\varepsilon}{3}, 1)$.
If $0 < |x-2| < \delta$, then $|x-2| < 1 \implies x+2 < 3$, and so

$$\left|x^2 - 4\right| = |(x-2)(x+2)| = |x-2|\,|x+2| < \delta \cdot 3 \le \varepsilon.$$

and so $\lim_{x\to 2} x^2 = 4$.
How did we come up with the choice of $\delta = \min(\frac{\varepsilon}{3}, 1)$? Scratch-work and creativity! Indeed it is far from the only suitable choice.

[1]The details are covered in the first two weeks of an Upper Division Analysis course...

No Limit In this picture the left- and right-limits are different, hence there is no limit at $x = a$. How can we view this in terms of ε and δ?



$\lim\limits_{x\to a^-} f(x) = L$ says that L is the only possible candidate for the limit.
Suppose we were given the indicated value ε. Regardless of our choice of $\delta > 0$, we will be able to find values of x (in the blue region) which satisfy both

$$0 < |x-a| < \delta \quad \text{and} \quad |f(x) - L| \geq \varepsilon$$

The definition of limit does not hold for all $\varepsilon > 0$, and so the limit does not exist.

Homework

1. Suppose that $\lim\limits_{x\to a} f(x) = L$. That is, for all given $\hat{\varepsilon} > 0$, there is some $\delta > 0$ for which

$$0 < |x-a| < \delta \implies |f(x) - L| < \hat{\varepsilon}.$$

Let $c \neq 0$ be constant and assume that $\varepsilon > 0$ is given. Show that there exists $\delta > 0$ for which

$$0 < |x-a| < \delta \implies |cf(x) - cL| < \varepsilon.$$

This proves that $\lim\limits_{x\to a} cf(x) = cL$. The other limit laws are proved similarly.