2.5 Continuity

Continuous functions behave nicely when taking limits.

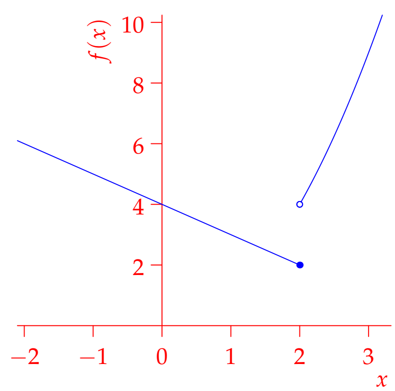
Definition. *f is* continuous at $x = a$ *if* $\lim_{x\to a} f(x) = f(a)$.
f is discontinuous at $x = a$ *if* $\lim_{x\to a} f(x) \neq f(a)$, *or does not exist: we call a a* discontinuity *of f.*
If f is continuous at all values a then we simply say that x is continuous.
If f is only defined when $x \geq a$ then we say that f is continuous at $x = a$ if $\lim_{x\to a^+} f(x) = f(a)$. The definition for $x \leq a$ is similar.[1]

Example $f(x) = x^3 - 2$ is continuous at all values $x = a$ since, by the limit laws, $\lim_{x\to a} x^3 - 2 = a^3 - 2 = f(a)$.

Discontinuities The loose idea of a continuous function is that one can draw its graph without taking one's pen off the paper. What then do functions with *discontinuities* look like? For a function to be continuous at $x = a$, we need three things:

1. $\lim_{x\to a} f(x)$ must exist.
2. $f(a)$ must exist.
3. $\lim_{x\to a} f(x)$ must equal $f(a)$.

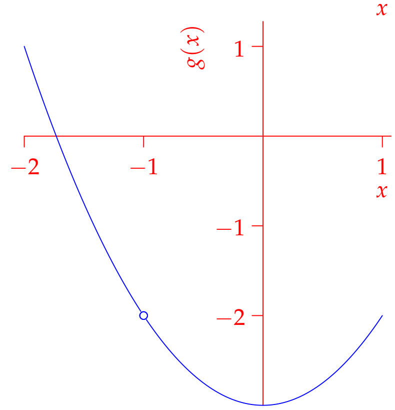
By thinking of examples which fail one each of these requirements, we can conjure several types of discontinuity.



Non-Example 1 Let $f(x) = \begin{cases} x^2 & \text{if } x > 2 \\ 4 - x & \text{if } x \leq 2 \end{cases}$
For this function, the left- and right- limits are distinct at $x = 2$:

$$\lim_{x\to 2^-} f(x) = 4 - 2 = 2$$
$$\lim_{x\to 2^+} f(x) = 2^2 = 4$$

and so $\lim_{x\to 2} f(x)$ does not exist. f is discontinuous at $x = 2$.



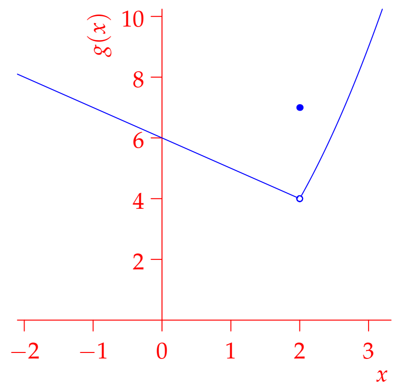
Non-Example 2 Let $g(x) = \dfrac{x^3 + x^2 - 3x - 3}{(x+1)} = \dfrac{(x^2-3)(x+1)}{(x+1)}$
In this case the limit at $x = -1$ exists: by the limit laws,

$$\lim_{x\to -1} g(x) = \lim_{x\to -1} x^2 - 3 = -4.$$

However $g(-1)$ does not exist and so -1 is a discontinuity of g.

[1] Strictly this is the definition of *right- and left-continuity.* We will not use these terms.

Non-Example 3 Let $h(x) = \begin{cases} x^2 - 2 & \text{if } x > 2 \\ 7 & \text{if } x = 2 \\ 6 - x & \text{if } x < 2 \end{cases}$



In this case the left- and right- limits are the same at $x = 2$:

$$\lim_{x \to 2^-} h(x) = 6 - 2 = 4$$
$$\lim_{x \to 2^+} h(x) = 2^2 = 4$$

and so $\lim_{x \to 2} h(x)$ exists and equals 4. This does not, however, equal $g(2) = 7$.

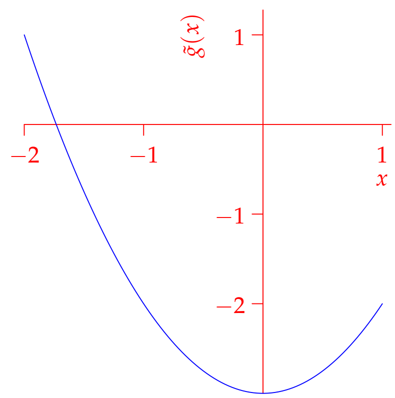
Removable Discontinuities Non-examples 2 and 3 are almost continuous in that a very small change to the function results in a continuous function.

Definition. *A discontinuity $x = a$ of f is* removable *if* $\lim_{x \to a} f(x) = L$ *exists.*

Theorem. *If a is a removable discontinuity of f then*

$$\tilde{f}(x) = \begin{cases} f(x) & x \neq a \\ L & x = a \end{cases} \quad \textit{is continuous at } a$$

We simply fill in a hole in the graph of the function so that $\tilde{f}$ may be drawn without taking the pen off the paper.



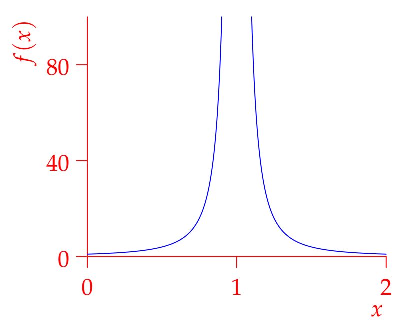
Non-example 2: the removable discontinuity has been removed.

Non-removable discontinuities If $\lim_{x \to a} f(x)$ is undefined, then $x = a$ is *non-removable*.

Examples

1. Non-example 1 has a non-removable discontinuity at $x = 2$.

2. $f(x) = \dfrac{1}{(x-1)^2}$ has $\lim_{x \to 1} f(x) = \infty$ which is undefined, so the discontinuity is non-removable. The vertical asymptote at $x = 1$ is an example of an *infinite discontinuity*.



3. The Sign function $\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ has a non-removable discontinuity at $x = 0$ since $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist. This is an example of a *jump discontinuity*.

There are Lots of Continuous Functions!

Theorem. *The following functions are continuous everywhere they are defined:*

1. *Polynomials*
2. *Rational functions*
3. *Trigonometric functions (note* $\tan x$ *has infinite discontinuities at* $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$*, etc.)*
4. *Power functions*
5. *Sums, products and quotients of continuous functions*

Example $f(x) = \cos x + 3x^2 + 17x^{3/2} - \dfrac{4x^{7/3} - 7}{x - 2}$ is continuous on the intervals $[0, 2)$ and $(2, \infty)$. Because of the $x^{3/2}$ term, f is undefined when $x < 0$. Dividing by $x - 2$ means that f has a vertical asymptote at $x = 2$.

We can also compose continuous functions

Theorem. *If* $\lim_{x \to a} g(x) = L$ *and* f *is continuous at* L*, then* $\lim_{x \to a} f(g(x)) = f(L)$*.*
In particular, if g *is continuous at* a *and* f *is continuous at* $g(a)$*, then* $f \circ g$ *is continuous at* a*.*

Examples

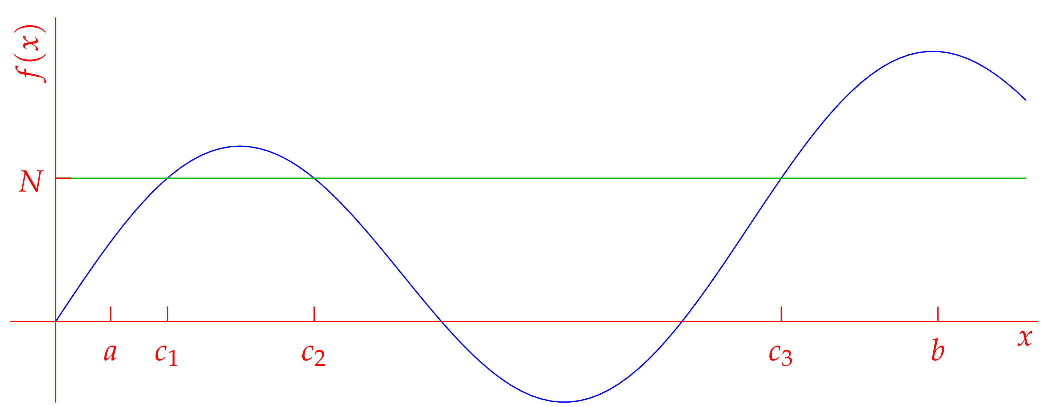
1. $f(x) = \sin(\sqrt{x^2 + 1})$ is continuous everywhere.
2. $\exp(x^2 - x^{-2})$ is continuous except at $x = 0$.
3. $g(x) = (1 - \cos x)^{-3}$ is continuous except when $\cos x = 1$, i.e. when $x \neq 0, \pm 2\pi, \ldots$

The Intermediate Value Theorem

Several powerful applications of continuous functions follow from this theorem.

Theorem (Intermediate Value)**.** *Suppose* f *is continuous on* $[a, b]$*, that* $f(a) \neq f(b)$*, and that* N *lies between* $f(a)$ *and* $f(b)$*. Then there is some value* c *between* a *and* b *for which* $f(c) = N$*.*

As the picture shows, there may be more than one choice of c



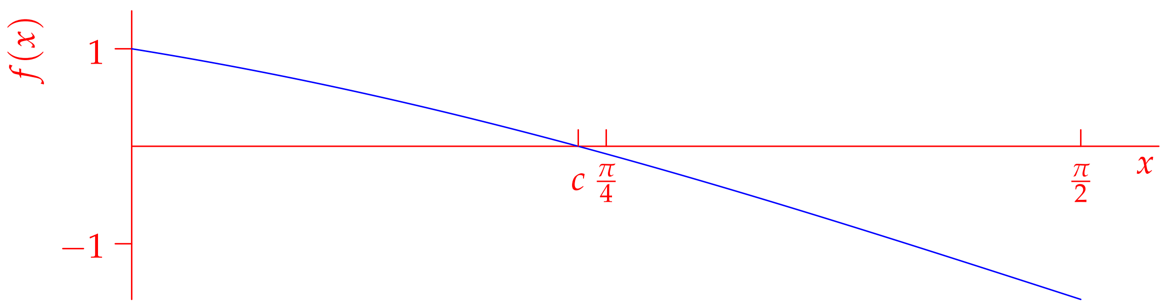
The Intermediate Value Theorem can be interpreted in many ways, for example:

- If the temperature at 8 a.m. is 65° F and at 12 p.m. is 88° F, then at some time during the morning, the temperature must have been 75° F.
- If you climb a 10,000 ft mountain from sea level, then at some point you must be at an elevation of 6,329 ft.
- Two runners run the same course in the opposite direction, starting at the same time at opposite ends and finishing at the same time. Then the runners must pass each other at some time during the race.

In mathematics, the Theorem is often used to show that certain equations have solutions, and to home in on these.

Corollary. *If $f(a)$ and $f(b)$ have opposite signs and f is continuous, then there is a solution c to $f(x) = 0$, which lies between a and b.*

Example $f(x) = \cos x - x$ is continuous, and $f(0) = 1$ and $f(\frac{\pi}{2}) = -\frac{\pi}{2}$ have opposite signs. Therefore the equation $x = \cos x$ has at least one solution $c \in (0, \frac{\pi}{2})$: that is, $\cos c = c$.
With a calculator, you can try more values of x, and use these to narrow down your estimate of c.



Homework

1. Show that $x^{17} + \tan(x) + 1 = 0$ has a solution near $x = 0$. Use your calculator to approximate it to 3 decimal places.
2. $f(x) = x + \frac{1}{x}$ satisfies $f(1) > 0$ and $f(-1) < 0$. Can we conclude that there is some value $c \in (-1, 1)$ for which $f(c) = 0$? Why/why not?