2.6 Limits at Infinity: Horizontal Asymptotes

We want to describe what happens to functions for very large x.

Definition. *Suppose that f has domain including (a, ∞) for some $a \in \mathbb{R}$. We write*

$$\lim_{x\to\infty} f(x) = L$$

if, as x gets unboundedly larger, the values of $f(x)$ get arbitrarily close[1] to L.

$\lim_{x\to-\infty} f(x) = L$ *is defined similarly.*

The line $y = L$ is a horizontal asymptote *of $y = f(x)$ if either*

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

A curve $y = f(x)$ necessarily has none, one, or two horizontal asymptotes.

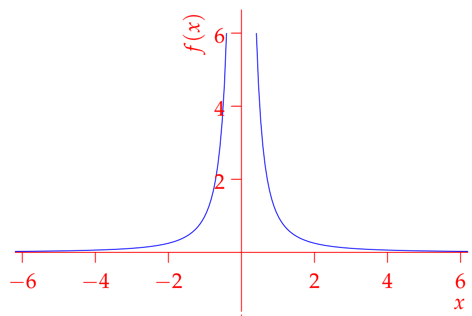
Limit Laws Most of the limit laws from Section 1.6 also apply to limits at infinity: for example, provided all three limits exist,

$$\lim_{x\to\infty} (f(x) + g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$$

Examples

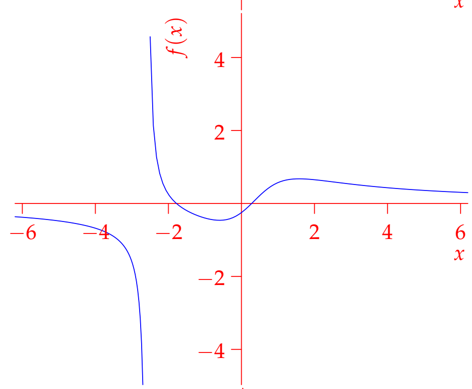
1. $f(x) = \dfrac{1}{x^2} \underset{x\to\pm\infty}{\longrightarrow} 0$

   More generally, $\frac{1}{x^r} \underset{x\to\pm\infty}{\longrightarrow} 0$ for $r > 0$



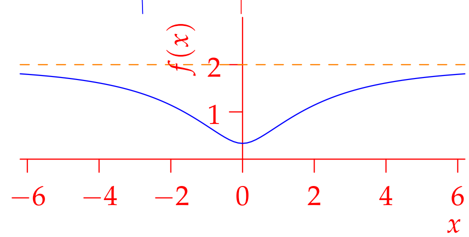
2. Dividing top and bottom by the highest power of x in the denominator can help compute limits:

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{2x^2+3x-1}{x^3+2x^2+4} &= \lim_{x\to\pm\infty} \frac{2x^{-1}+3x^{-2}-x^{-3}}{1+2x^{-1}+4x^{-3}} \\ &= \frac{\lim_{x\to\pm\infty}(2x^{-1}+3x^{-2}-x^{-3})}{\lim_{x\to\pm\infty}(1+2x^{-1}+4x^{-3})} \\ &= 0\end{aligned}$$



3. Square-roots are continuous, so we can take the limit operator inside...

$$\begin{aligned}\lim_{x\to\pm\infty} \frac{\sqrt{4x^2+1}}{\sqrt{x^2+9}} &= \sqrt{\lim_{x\to\pm\infty} \frac{4x^2+1}{x^2+9}} \\ &= \sqrt{\frac{\lim_{x\to\pm\infty} 4+x^{-2}}{\lim_{x\to\pm\infty} 1+9x^{-2}}} = \frac{\sqrt{4}}{\sqrt{1}} = 2\end{aligned}$$

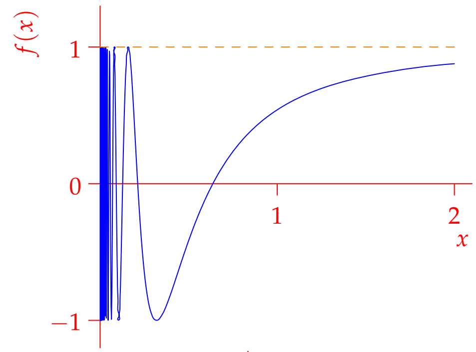


[1] The strict definition is non-examinable: For all $\varepsilon > 0$ there exists N such that $x > N \implies |f(x) - L| < \varepsilon$.

Examples

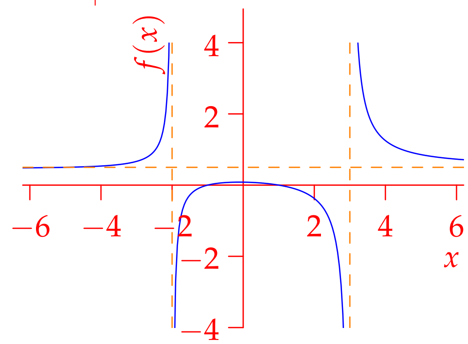
4. Cosine is continuous, therefore,

$$= \lim_{x\to\pm\infty} \cos\left(\frac{1}{x}\right) = \cos\left(\lim_{x\to\pm\infty} \frac{1}{x}\right)$$
$$= \cos 0 = 1$$



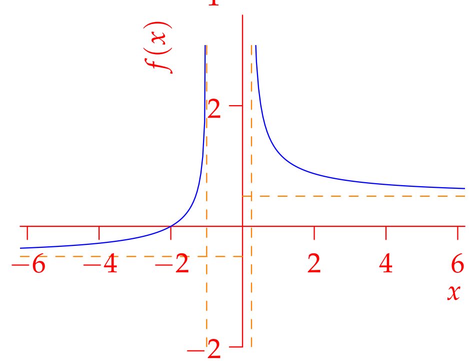
5. $f(x) = \dfrac{x^2-1}{2x^2-2x-12} = \dfrac{(x+1)(x-1)}{2(x-3)(x+2)}$ has one horizontal and two vertical asymptotes:

$$\lim_{x\to\pm\infty} \frac{x^2-1}{2x^2-2x-12} = \frac{1}{2}$$
$$\lim_{x\to-2^\pm} \frac{x^2-1}{2x^2-2x-12} = \mp\infty$$
$$\lim_{x\to3^\pm} \frac{x^2-1}{2x^2-2x-12} = \pm\infty$$



6. $f(x) = \dfrac{x+2}{\sqrt{4x^2+3x-1}} = \dfrac{x+2}{\sqrt{(4x-1)(x+1)}}$ has two horizontal and two vertical asymptotes:

$$\lim_{x\to\infty} f(x) = \frac{1}{2} \qquad \lim_{x\to-\infty} f(x) = -\frac{1}{2}$$
$$\lim_{x\to-1^-} f(x) = \infty \qquad \lim_{x\to\frac{1}{4}^+} f(x) = \infty$$



Example 6 can cause difficulties: remember that $\sqrt{x^2} = |x|$, so, if $x \neq 0$ we have

$$\frac{x+2}{\sqrt{4x^2+3x-1}} = \frac{x(1+1/x)}{|x|\sqrt{4+3/x-1/x^2}} \implies \lim_{x\to\infty} f(x) = \lim_{x\to\pm\infty} \frac{x}{|x|\sqrt{4}} = \pm\frac{1}{2}$$

Infinite Limits at Infinity

Definition. *Suppose that, as x gets unboundedly larger, so do the values of $f(x)$. We write*[2]

$$\lim_{x\to\infty} f(x) = \infty$$

$\lim_{x\to\infty} f(x) = -\infty$ *and* $\lim_{x\to-\infty} f(x) = \pm\infty$ *are similar.*

For rational functions we can again use the procedure of dividing numerator and denominator by the highest power of x in the denominator.

[2]Again the strict definition is non-examinable: For all $M > 0$ there exists $N > 0$ such that $x > N \implies f(x) > M$.

Example $\displaystyle\lim_{x\to-\infty}\frac{x^2-x}{2x+1}=\frac{\lim_{x\to-\infty}(x-x^{-1})}{\lim_{x\to-\infty}(2+x^{-1})}=\frac{-\infty}{2}=-\infty.$

It is easy to get confused when calculating with infinite limits, so take your time.

Example In the following, the right hand side is meaningless:

$$\lim_{x\to\infty}3x-x^2\neq\lim_{x\to\infty}3x-\lim_{x\to\infty}x^2=\infty-\infty$$

Instead we must factorize:

$$\lim_{x\to\infty}3x-x^2=\lim_{x\to\infty}x(3-x)=-\infty$$

since x increases and $3-x$ descreases without bound.

Homework

1. (a) The *hyperbolic tangent* function is defined by $\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$. Find its horizontal asymptotes.

   (b) Suppose that $x>y$. Prove that $\tanh x>\tanh y$ (*can you do this algebraically, that is without using any derivatives?*). Use this to help sketch the graph of tanh.

2. Sketch the graphs of $y=e^{x^2}$ and $y=e^{1/x^2}$. Check for horizontal asympototes.