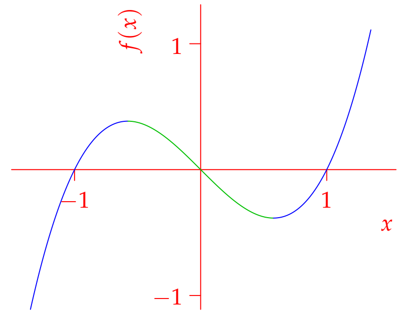
2.7 Derivatives and Rates of Change



Differentiation is the process of calculating and analyzing the rate of change of a function. By looking at a graph, we can say *qualitative* things about the rate of change of a function. For example in the picture, as x increases, the value of $f(x) = x^3 - x$ is alternately increasing, decreasing, and increasing.

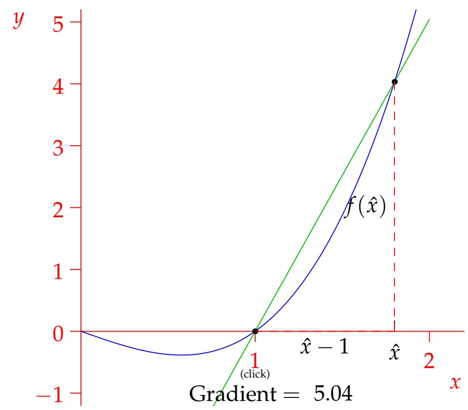
The question for a mathematician is how to *quantify* this? Before we do that, we need to agree what we mean by *rate of change*.

Definition. *The* rate of change *of a function $f(x)$ at $x = a$ is the slope of the tangent line to the graph $y = f(x)$ at $x = a$, if such a tangent line exists.*

Recall from Section 2.1 how we compute the tangent line at a point.

Example For $f(x) = x^3 - x$ at $(1, 0)$ we construct secant lines through $(1, 0)$ and $(\hat{x}, \hat{x}^3 - \hat{x})$ and compute the limit of their slopes:

$$\begin{aligned} m_{\hat{x}} &= \frac{f(\hat{x}) - f(1)}{\hat{x} - 1} = \frac{\hat{x}^3 - \hat{x}}{\hat{x} - 1} \\ &= \frac{(\hat{x} - 1)(\hat{x}^2 + \hat{x})}{\hat{x} - 1} \\ &= \hat{x}^2 + \hat{x} \\ \implies m &= \lim_{\hat{x} \to 1} m_{\hat{x}} = 2 \end{aligned}$$



The *rate of change* of the function $f(x) = x^3 - x$ at $x = 1$ is therefore 2.

Tangent Lines in the Abstract Given a general curve $y = f(x)$ we follow the same procedure.

Definition. *The* tangent line *to the curve $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ with slope*

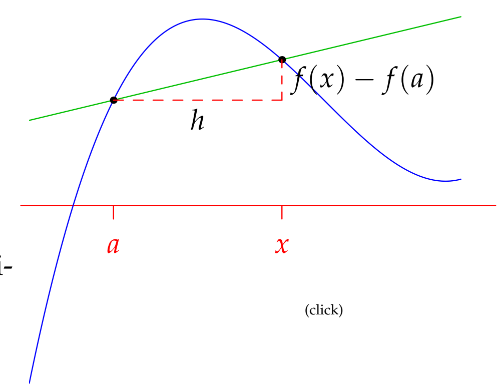
$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

supposing the limit exists. The tangent line has equation

$$y = m(x - a) + f(a)$$

Often we think of $h = x - a$ as being important, and the definition becomes

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$



Since 'slope of the tangent line' is such a mouthful, we have a special term…

Definition 2.1. A function f is *differentiable* at $x = a$ if the above limits

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exist. If f is differentiable at $x = a$, then this limit is denoted $f'(a)$ or $\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a}$ and is termed the *derivative of f at $x = a$.*

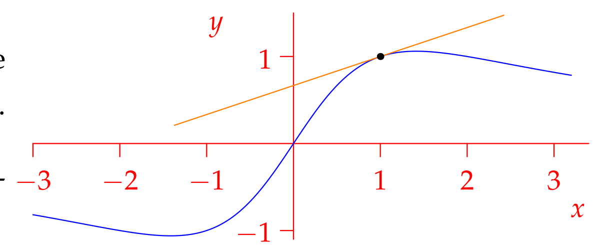
Examples

1. If $f(x) = x^2$, then

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2-3^2}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h}$$
$$= \lim_{h\to 0} 6+h = 6$$

hence f is differentiable at $x = 3$ with derivative $f'(3) = 6$.

2. Show that the function $f(x) = \dfrac{3x}{x^2+2}$ is differentiable at $x = 1$, and compute the equation of its tangent line.



First compute the limit: a little algebraic simplification is required.

$$\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{\frac{3x}{x^2+2}-1}{x-1} = \lim_{x\to 1} \frac{3x-x^2-2}{(x-1)(x^2+2)}$$
$$= \lim_{x\to 1} \frac{(2-x)(x-1)}{(x-1)(x^2+2)} = \lim_{x\to 1} \frac{2-x}{(x^2+2)} = \frac{1}{3}$$

Thus f is differentiable at $x = 1$ with derivative $f'(1) = \frac{1}{3}$. The tangent line therefore has gradient $\frac{1}{3}$ and passes through the point $(1, f(1)) = (1, 1)$. Its equation is then

$$y = m(x-1) + 1 = \frac{1}{3}(x-1) + 1 = \frac{1}{3}(x+2)$$

We could alternatively have computed $\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$.

Leibniz notation and rates of change The two most famous contributors to Calculus, Issac Newton and Gottfried Wilhelm Leibniz, had different notations for derivative. The $f'(a)$ notation is a modification of Newton's approach,[1] while $\frac{\mathrm{d}f}{\mathrm{d}x}$ is Leibniz's notation. The importance of Leibniz's notation is that it reminds us what derivatives are: rates of change of one quantity with respect to another. The units of a derivative should then be obvious in any situation. For example:

[1] In fact Newton used a dot over a variable and always differentiated with respect to time, so if $y = f(t)$, the derivative of f with respect to t would be denoted $\dot{y}$.

Velocity is the rate of change of *position* with respect to time. If a particle is at distance $s(t)$ from a fixed point at time t, then the average velocity of the particle between times $t = a$ and $t = b$ is

$$v_{av} = \frac{\text{distance}}{\text{time}} = \frac{s(b) - s(a)}{b - a}$$

The *instantaneous velocity* at $t = a$ is the derivative

$$v(a) = \left.\frac{\mathrm{d}s}{\mathrm{d}t}\right|_{t=a} = \lim_{b\to a} \frac{s(b) - s(a)}{b - a} = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h}$$

The notation helps us get units right: velocity has units of distance divided by time, e.g. m/s, mph, ft/s, miles/year, furlongs per fortnight, etc.

Electric Current is the rate of change of charge with respect to time. For example, suppose that the charge Q coulombs stored in a capacitor at time t seconds is

$$Q(t) = 10 - \frac{10}{1+t}$$

The current flow at $t = 3$ seconds is then

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \lim_{h\to 0} \frac{Q(3+h) - Q(3)}{h} = \frac{5}{8}\,\mathrm{C/s}.$$

This unit, coulombs per second, is usually called an *ampere.*

Marginal Profit (economics) Suppose that a tea seller selling x lb of tea makes \$ $p(x)$ profit. If the tea seller is currently selling 100 lb of tea and intends to sell a small amount Δx lb *more* tea is sold, then the increase in profit will be

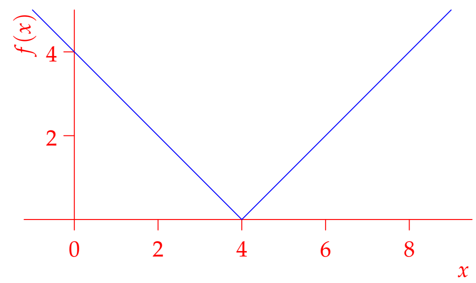
$$\Delta p = p(100 + \Delta x) - p(100)$$

The instantaneous rate of change of p is the derivative:

$$\lim_{\Delta x\to 0} \frac{\Delta p}{\Delta x} = p'(100)$$

measured in dollars per pound. If, for instance, $p'(100) = 0.7$ \$/lb, then selling 101 lb of tea will yield approximately 70 cents more profit than selling 100 lb.

A Function with no Rate of Change Let $f(x) = |x-4|$. What happens if we try to differentiate this at $x = 4$? We are obliged to calculate the limit



$$\lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

However,

$$\frac{|h|}{h} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$$

whence the left- and right-limits of $\frac{|h|}{h}$ are non-equal and the limit does not exist. If you think about why, the function $f(x)$ has a *corner* at $x = 4$. Is the function still decreasing, or is it increasing, or neither? Hopefully, you agree that the very idea of rate of change makes no sense for this function at $x = 4$: we say that f is *not differentiable* at $x = 4$.

Homework

A company sells cellphone plans. The cost of data is priced dependent on how much you use: if you use x megabytes of data per month, the cost in dollars will be

$$c(x) = x + 10\left(1 - \frac{1}{1+x^2}\right)$$

1. What are the units of the rate of change $c'(a)$?
2. *Use the limit definition* to compute the value of $c'(a)$ for all positive values of a.
3. You should find that $c'(a) > 1$ for all $a > 0$. Interpret what this means in terms of the cost of an additional megabyte of data.