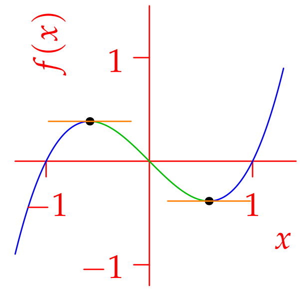
2.8 The Derivative as a Function

Typically, we can find the derivative of a function f at many points of its domain:

Definition. *Suppose that f is a function which is differentiable at every point x of an open interval (a, b). Its* derivative *is the function*

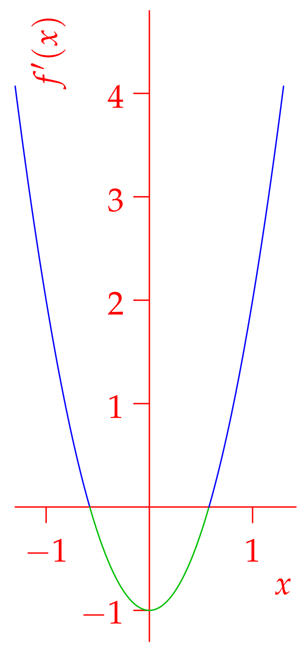
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of f' must include the interval (a, b).



A function and its derivative are drawn:

f *is increasing* $\iff f' > 0$
f *is decreasing* $\iff f' < 0$
f *has a horizontal tangent line* $\iff f' = 0$



Example If $f(x) = x^3 - x$, then its derivative is

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h} \\ &= \lim_{h \to 0} \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - x - h] - [x^3 - x]}{h} \\ &= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 - 1 \\ &= 3x^2 - 1 \end{aligned}$$

The domains of both f and f' are the real line $\mathbb{R}$. Note how the graphs correspond: when f is increasing, the derivative is positive, when f is decreasing, the derivative is negative.

We may *compute* similarly for many other functions.[1] You should draw the graphs of these: do the graphs fit with your calculations?

1. $f(x) = \frac{1}{x} \implies f'(x) = -\frac{1}{x^2}$ both with domain $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$.
2. $f(x) = \sqrt{x} \implies f'(x) = \frac{1}{2\sqrt{x}}$. The former has $\text{dom}(f) = [0, \infty)$ while the latter has $\text{dom}(f') = (0, \infty)$.
3. Draw the graph of $y = \sin x$. Sketch underneath the graph of its derivative, just by thinking about when $\sin x$ is increasing and where it is decreasing. The new graph should look very familiar...

[1] For now this means *using the limit definition*. Nice formulæ such as the power law will have to wait until *after* the midterm...

Notation

If $y = f(x)$ there are many notations for the derivative function:

$$f'(x),\ \frac{\mathrm{d}y}{\mathrm{d}x},\ \frac{\mathrm{d}f}{\mathrm{d}x},\ \frac{\mathrm{d}}{\mathrm{d}x}f(x),\ y',\ Df(x),\ D_x f(x)$$

The *value* of the derivative function at $x = a$ is denoted

$$f'(a),\ \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=a},\ \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a},\ y'(a),\ Df(a),\ D_x f(a)$$

The symbol $\frac{\mathrm{d}}{\mathrm{d}x}$ may be thought of as an *operator:* turning a function into its derivative.

For example, if $y = f(x) = \sqrt{x}$ then we know that $f'(x) = \frac{1}{2\sqrt{x}}$. We could instead write $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$, or $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{1}{2\sqrt{x}}$. Moreover $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$ and $\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=9} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$.

Higher Derivatives

We can differentiate derivatives! For example, the *second derivative* of f is the derivative of f': that is

$$f''(x) = \lim_{h\to 0}\frac{f'(x+h) - f'(x)}{h}$$

if the limit exists. Leibniz's alternative notation for second derivatives reads as if one is *squaring the derivative operator:*

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}f}{\mathrm{d}x} = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 f$$

This can help when trying to understand units. We can similarly compute higher order derivatives:

Third: $f'''(x) = \frac{\mathrm{d}^3 f}{\mathrm{d}x^3}$

Fourth: $f^{(4)}(x) = \frac{\mathrm{d}^4 f}{\mathrm{d}x^4}$

Fifth: $f^{(5)}(x) = \frac{\mathrm{d}^5 f}{\mathrm{d}x^5}$

and so on. The bracket notation $f^{(n)}(x)$ is preferred for derivatives higher than third because of the increased difficulty counting multiple prime symbols $'$.

Example $f(x) = 3x^2 + 2x$ has $f'(x) = 6x + 2$ and $f''(x) = 6$. Then $f^{(n)}(x) = 0$ for all $n \geq 3$.

Acceleration When $s(t)$ is the distance traveled by a particle at time t, the derivative $v(t) = s'(t)$ is the particle's *velocity.* The second derivative $a(t) = s''(t) = v'(t)$ is the *acceleration* of the particle.[2]

Units: remember that each differentiation appends a 'per unit time' to the units. Acceleration is therefore measured as "distance-per-time-per-time:" for example,

$\mathrm{m/s^2} = \mathrm{ms^{-2}}$ = meters per second per second

$\mathrm{ft/hr^2} = \mathrm{ft\,hr^{-2}}$ = feet per hour per hour

[2] In this context, the third derivative s''' is referred to as the *jerk.*

Example After t seconds, a ball has height $s(t) = 1 + 20t - 4.9t^2$ meters.
Its velocity is $v(t) = s'(t) = 20 - 9.8t$ m/s.
Its acceleration is $a(t) = s''(t) = -9.8$ m/s^2. Note that this last is the gravitational constant.

What does a differentiable function look like? So much for calculating with limits. We want an intuitive idea[3] of what to expect from the graphs of differentiable and non-differentiable functions. Similarly to how we understood the concept of *continuity*, we consider all the ways in which a function might *fail* to be differentiable. The most obvious way turns out to be related to continuity!

Theorem. *If f is differentiable at $x = a$, then f is continuous at $x = a$.*

Proof. Suppose that f differentiable at $x = a$. If $x \neq a$, then

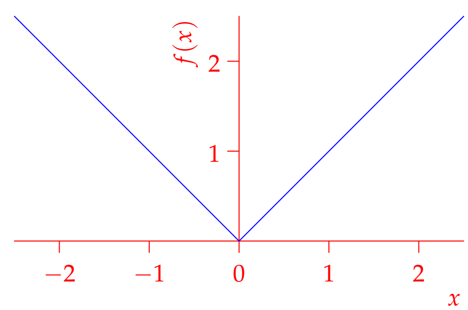
$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a)$$
$$\implies \lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}(x - a) = f'(a) \cdot 0 = 0$$

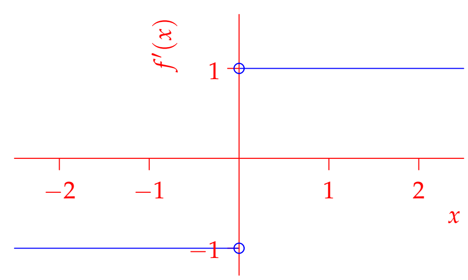
Therefore f is continuous at a. ∎

An equivalent statement of the Theorem is:

If f is *discontinuous* at $x = a$ then f is *non-differentiable* at $x = a$.

Thus differentiable functions can be drawn without taking your pen off the page. The converse to this is false however. It is possible for a function to be continuous but non-differentiable. For this to happen, the limit $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ cannot exist. There are two common possibilities.[4]





1. *Corners* For example $f(x) = |x|$ is continuous at $x = 0$. What about its derivative? If $x \neq 0$, the derivative is

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

To find $f'(0)$ we would need to calculate

$$\lim_{h \to 0} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

which does not exist.
Therefore $|x|$ is continuous at $x = 0$, but *not* differentiable.

[3]Similarly to how a continuous function should be drawable without taking your pen off the page.
[4]There are more esoteric examples, such as the blancmange curve which is continuous everywhere and differentiatiable nowhere, but such things are well-beyond the scope of this course!

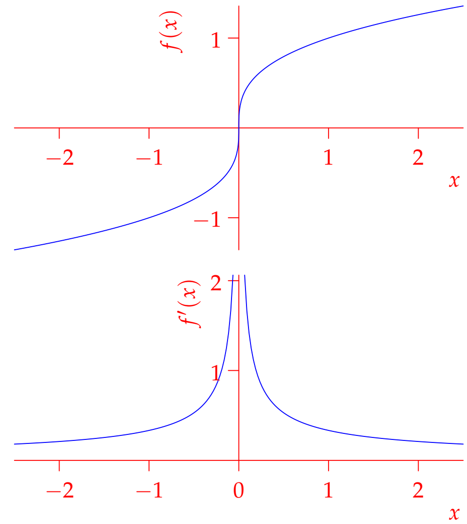
2. *Vertical tangents* For example, $f(x) = \sqrt[3]{x} = x^{1/3}$ is continuous everywhere. If we want to search for a derivative at $x = 0$ we must compute the limit

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h^{-2/3} = +\infty$$

By computing limits we can see that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{3x^{2/3}}$$

provided $x \neq 0$. Thus f is continuous at $x = 0$, but not differentiable at $x = 0$: it has a vertical tangent line.

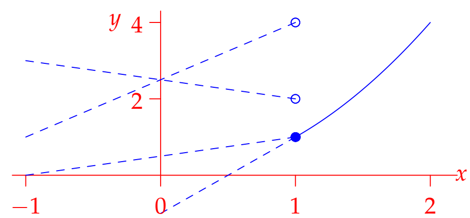


Example: Choosing to make a function differentiable Find constants a, b such that

$$f(x) = \begin{cases} a + bx & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$$

is differentiable for all x.

As the possible graphs of f show, to the left of $x = 1$ the function is a straight line. Which choice of line will make the function differentiable at $x = 1$?



We can answer this in words: firstly, a differentiable function must be continuous, so the straight line we choose must pass through the point $(1, f(1)) = (1, 1)$. Secondly, a differentiable function must ave the same rate of change when calculated as a left- or a right-limit, whence the required straight line must have the same slope as $y = x^2$ as $x = 1$. Now we calculate:

Continuity at $x = 1$: We require

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1) \implies a + b = 1$$

Any function f with $a + b = 1$ will have the straight line intersecting the parabola at $(1, 1)$.

Differentiability at $x = 1$: We require

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \implies b = 2$$

Putting these together, we see that f is differentiable if and only if $a = -1$ and $b = 2$. Indeed its derivative is

$$f'(x) = \begin{cases} 2 & \text{if } x < 1 \\ 2x & \text{if } x > 1 \end{cases}$$

Homework

1. Compute all of the derivatives not explicitly found above: use the limit definition!

2. Let $f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

   (a) Calculate $f'(x)$ for $f(x) = x\,|x|$.

   (b) What about $f''(x)$? For what values of x does this make sense?

   (c) Can you guess a formula for a function which is twice-differentiable at every value of x but not three-times differentiable everywhere? Compute its first, second and third derivatives.

3. The *binomial theorem* states that if n is a positive integer, then

$$(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^k h^{n-k} = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n$$

   where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ is the *binomial coefficient.*
   Use this to *prove* the power law for differentiation. If n is a positive integer, then

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$