3 Differentiation Rules

3.1 Derivatives of Polynomials and Exponential Functions

Since derivatives are so useful, we need to be able to compute more efficiently than just using the limit definition.

In this section we gather several results that help us differentiate complex functions quickly.

Theorem. *If $f(x) = c$ is constant, then $f'(x) = 0$. This can also be written* $\frac{\mathrm{d}}{\mathrm{d}x}c = 0$.

Proof. Here are two possible arguments:

1. The graph is a horizontal line with zero slope, and so $f'(x) = 0$.
2. $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$ ∎

The Power Law By now, you should have spotted a pattern:

$$\frac{\mathrm{d}}{\mathrm{d}x}x = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2, \qquad \frac{\mathrm{d}}{\mathrm{d}x}x^4 = 4x^3, \quad \text{etc.}$$

In general, this becomes:

Theorem (Power Law). *Let $f(x) = x^n$ where n is a constant real number. Then its derivative is*[1]

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$

Examples

1. $\frac{\mathrm{d}}{\mathrm{d}x}x^{17} = 17x^{17-1} = 17x^{16}$ (differentiable on $\mathbb{R}$)
2. $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{\mathrm{d}}{\mathrm{d}x}x^{1/2} = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ (differentiable on $(0, \infty)$)
3. $\frac{\mathrm{d}}{\mathrm{d}x}x^{-4/3} = -\frac{4}{3}x^{-4/3-1} = -\frac{4}{3}x^{-7/3}$ (differentiable on $(-\infty, 0) \cup (0, \infty)$)
4. $\frac{\mathrm{d}}{\mathrm{d}x}1 = \frac{\mathrm{d}}{\mathrm{d}x}x^0 = 0$ follows the pattern of the power law
5. $\frac{\mathrm{d}}{\mathrm{d}x}x^{3.2964} = 3.2964x^{2.2964}$ (differentiable on $(0, \infty)$)
6. $\frac{\mathrm{d}}{\mathrm{d}x}x^{\pi} = \pi x^{\pi-1}$ (differentiable on $(0, \infty)$)

[1] *Where* f is differentiable depends on n: the domain of f' is either $\mathbb{R}$, or the non-zero reals $(-\infty, 0) \cup (0, \infty)$, just the interval $(0, \infty)$.

Theorem (Linearity). $\frac{\mathrm{d}}{\mathrm{d}x}(af(x) + bg(x)) = af'(x) + bg'(x)$

All these results need to be *proved* from the limit definition.

Example proof. Suppose that f and g are differentiable at $x = a$. Then

$$\begin{aligned}
\lim_{h \to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} &= \lim_{h \to 0} \frac{[f(a+h) + g(a+h)] - [f(a) + g(a)]}{h} \\
&= \lim_{h \to 0} \frac{(f(a+h) - f(a)) + (g(a+h) - g(a))}{h} \\
&= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \\
&= f'(a) + g'(a)
\end{aligned}$$

Hence $f + g$ is differentiable at $x = a$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) + g(x)) = f'(x) + g'(x)$$

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Linearity together with the power law allows us to differentiate any polynomial:

Examples

1. $\frac{\mathrm{d}}{\mathrm{d}x}(7x - 3x^5) = \frac{\mathrm{d}}{\mathrm{d}x}(7x) - \frac{\mathrm{d}}{\mathrm{d}x}(3x^5) = 7\frac{\mathrm{d}}{\mathrm{d}x}x - 3\frac{\mathrm{d}}{\mathrm{d}x}x^5 = 7 \cdot 1 - 3 \cdot 5x^4 = 7 - 15x^4$
2. $f(t) = 1 + 20t - 4.9t^2$, then

   $$f'(t) = 20 - 9.8t, \quad \text{and} \quad f''(t) = -9.8$$

3. For what value of c is $y = \frac{3}{2}x + 6$ tangent to the curve $y = c\sqrt{x}$? What is the point of tangency?

   We require two things: if the two curves are tangent at x, then:

   The curves must meet at x That is $\frac{3}{2}x + 6 = c\sqrt{x}$.
   The curves must have the same slope at x Otherwise said, their derivatives must be equal:

   $$\frac{3}{2} = \frac{1}{2}cx^{-1/2}$$

   The second equation can be re-written $c = 3\sqrt{x}$. We may substitute this into the first equation to obtain

   $$\frac{3}{2}x + 6 = 3\sqrt{x}\sqrt{x} = 3x \implies x = 4$$

   We can now substitute into the second equation to solve for $c = 3\sqrt{4} = 6$.
   The point of tangency is $(4, 6\sqrt{4}) = (4, 12)$.

Exponential Functions

Let $f(x) = a^x$, then

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h} = a^x f'(0)$$

provided the limit exists. Therefore the slope of an exponential function is proportional to original function. Recall that e was *defined* as the number which makes the proportionality constant equal to 1. We therefore have:

Theorem. $\lim\limits_{h \to 0} \frac{e^h - 1}{h} = 1$. *Moreover, $f(x) = e^x$ is differentiable, with derivative*

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$$

Example For what value of x are the curves $y = 2e^x$ and $y = 1 + 3x$ parallel?

Recall that *parallel* means 'have the same slope'. We must therefore differentiate both functions and equate the derivatives:

$$2e^x = 3 \iff x = \ln \frac{3}{2} \approx 0.4055$$