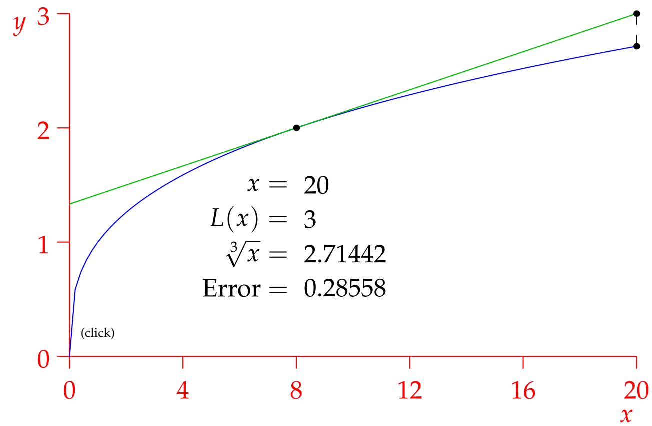
3.10 Linear Approximations and Differentials

In the picture, the tangent line to $y = x^{1/3}$ at $x = 8$ is viewed as an *approximation* to the original curve.

$y = L(x)$ is the equation of the tangent line.

The *error* is the difference $L(x) - x^{1/3}$ between the approximate and correct values, shown correct to 5 d.p.



Theorem. *Suppose that $y = f(x)$ is a differentiable curve at $x = a$. Then the tangent line at $x = a$ has equation*

$$y = f(a) + f'(a)(x - a)$$

We call the above equation the *linear approximation* or *linearization* of $y = f(x)$ at the point $(a, f(a))$ and write

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

We sometimes write $L_a(x)$ to stress that the approximation is *near a.*

Example Consider the cube root function above: $y = f(x) = \sqrt[3]{x} = x^{1/3}$. We approximate near $x = 8$.
We have

$$f(8) = 2, \text{ and } f'(x) = \frac{1}{3}x^{-2/3} \implies f'(8) = \frac{1}{12}$$

whence the linear approximation is

$$L_8(x) = f(8) + f'(8)(x - 8) = 2 + \frac{1}{12}(x - 8)$$

This can be used, for example, to approximate cube roots without using a calculator: e.g.

$$\sqrt[3]{8.1} \approx 2 + \frac{1}{120} = 2.0083\dot{3}.$$

Example The natural exponential function $f(x) = e^x$ has linear approximation $L_0(x) = 1 + x$ at $x = 0$. It follows that, for example, $e^{0.2} \approx 1.2$. The exact value is 1.2214 to 4 d.p.

Localism The linear approximation is only useful *locally*: the approximation $f(x) \approx L_a(x)$ will be good when x is close to a, and typically gets *worse* as x moves away from a. For large differences between x and a, the approximation $L_a(x)$ will be essentially useless. The challenge is that the quality of the approximation depends hugely on the function f.

Example Find an approximation to $\sqrt{15}$.

Since $\sqrt{16} = 4$ is easy to compute and 16 is close to 15, we consider the linear approximation to $f(x) = \sqrt{x}$ centered at $x = 16$. First differentiate:

$$f'(x) = \frac{1}{2}x^{-1/2} \implies f'(16) = \frac{1}{2 \cdot 4} = \frac{1}{8}$$

Therefore

$$L_{16}(x) = f(16) + f'(16)(x - 16) = 4 + \frac{1}{8}(x - 16)$$

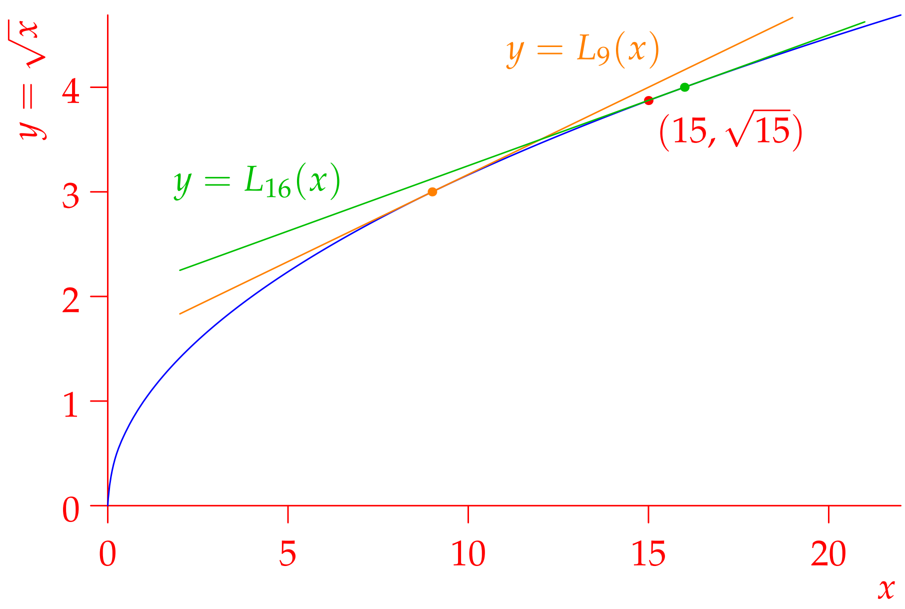
It follows that

$$\sqrt{15} \approx L_{16}(15) = 4 - \frac{1}{8} = 3\frac{7}{8} = \frac{31}{8} = 3.875$$

We could instead have used the linear approximation centered at $x = 9$, also a nice value for the square-root function. In this case we obtain

$$L_9(x) = f(9) + f'(9)(x - 9) = 3 + \frac{1}{6}(x - 9) \implies \sqrt{15} \approx L_9(15) = 4$$

Since 15 is much closer to 16 than to 9, we expect that the approximation 3.875 is the superior estimate. Indeed, if you ask your calculator, you'll find that $\sqrt{15} = 3.873$ to 3 d.p., which backs up the picture below.



Errors The *error* in an approximation $f(x) \approx L_a(x)$ is the difference $\mathcal{E}_a(x) = L_a(x) - f(x)$. In the above example, the errors using the two approximations, to 3 d.p. are

$$\mathcal{E}_9(15) = 0.027, \qquad \text{and} \qquad \mathcal{E}_{16}(15) = 0.002$$

Clearly an error closer to zero means a better approximation.

Differentials

Comparing the two notations for dervative, we are used to writing $f'(x) = \frac{dy}{dx}$. Recall the motivation for Leibniz's notation:

$$\left.\frac{dy}{dx}\right|_{x=a} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where we are treating $\Delta x = x - a$ as a small change in the value of x which *induces*, via the function f, a corresponding change $\Delta y = f(x) - f(a)$ in the value of f. If we view dx and dy as *infinitessimally small changes* in x, y, we may write

$$f'(x) = \frac{dy}{dx} \implies dy = f'(x)\,dx$$

What does this mean? If $x = a$ and we increase x by an infinitessimally small amount dx, then y will increase by an infinitessimally small amount $dy = f'(a)\,dx$.

Definition. dx *and* dy *are termed* differentials.

Differentials are useful when the *value* of a quantity is unimportant, only the *approximate change* in the quantity in response to a change in input is desired. As long as the change dx in input x is very small, the differential dy will be a good approximation to the expected change in the output y.

Example A car company selling x cars per month has the following model for the profit ($) made

$$p(x) = \frac{1}{10}x^3\left[1 - \left(\frac{x}{500}\right)^2\right]$$

Suppose that the company is currently selling 100 cars per month. If, in the next month 103 cars are sold, what will be the approximate change in the profit?

Here $p(x) = \frac{1}{10}x^3 - \frac{1}{10\cdot 500^2}x^5$, whence

$$p'(x) = \frac{3}{10}x^2 - \frac{1}{2\cdot 500^2}x^4 \implies p'(100) = 3000 - \frac{1}{50}\cdot 100^2 = 2800$$

If the increase in car sales is $dx = 3$, then the approximate increase in profits is

$$dp = p'(100)\,dx = 2800 \cdot 3 = \$\,8,400$$

Computing the precise change in profits is not too difficult, but it is time consuming.

$$p(103) - p(100) = 104,635.60 - 96,000 = \$\,8,635.60$$

One advantage of the differential method is that we can easily compute approximations to other possible outcomes. For instance, if the company sells 98 cars, then

$$dp = p'(100)\,dx = 2800 \cdot (-2) = -\$\,5,600$$

110 cars will yield approximately $dp = \$\,28,000$ more profit. Of course these approximations get worse the further from $x = 100$ we get.[1]

[1] $p(110) - p(100) = \$\,30657.96$ exactly.

Example: painting a surface Suppose you wish to paint the outside surface of a cylindrical tube and you want to estimate how much paint is needed. The tube has a length of $\ell = 10\,\text{cm}$ and a radius of $r = 3\,\text{cm}$. Suppose that the paint is to be applied to a thickness of 1 mm. What volume of paint, approximately, is required.

We know that the volume of a cylinder of radius r and length ℓ is $V = \pi r^2 \ell$. Painting the cylinder to a thickness of 1 mm is equivalent to *increasing* the radius of the cylinder by 1 mm. The paint required will be the consequent increase in volume. Since $\ell = 10$ is constant for this problem, we view V as a function of r and differentiate:

$$V'(r) = 2\pi\ell r \implies \mathrm{d}V = 2\pi\ell r\,\mathrm{d}r$$

The thickness of the paint is the increas in radius $\mathrm{d}r = 1$ mm=0.1 cm, whence the required volume of paint is approximately

$$\mathrm{d}V = 2\pi \cdot 10 \cdot 3 \cdot 0.1 = 6\pi = 18.85\,\text{cm}^3, \text{ to 2 d.p.}$$

The exact value in this case would be $V(3.1) - V(3) = 6.1\pi \approx 19.16$.

Errors

Differentials can also be used to estimate the error in a quantity. Suppose that $y = f(x)$, where the value of x is known within some error range $x \pm \mathrm{d}x$. The resulting potential error in y is computed using the differential $\mathrm{d}y = f'(x)\,\mathrm{d}x$.

Example The side length of a cube is measured using a ruler and observed to be $x = 10$ cm. The volume of the cube is therefore $1000\,\text{cm}^3$.
However the ruler is only marked every millimeter, so it might be reasonable to say that the potential error in the measurement x is $\mathrm{d}x = \frac{1}{2}$ mm= $\frac{1}{20}$ cm. What is the resulting potential error in the volume?

Since $V(x) = x^3$ we see that

$$\mathrm{d}V = 3x^2\,\mathrm{d}x = 3 \cdot 10^2 \cdot \frac{1}{20} = 15\,\text{cm}^3$$

It might therefore be appropriate to state the volume of the cube as $V = 1000 \pm 15\,\text{cm}^3$.
Compare this with $V(10.05) = 1015.075$ and $V(9.95) = 985.075$.

Percentage and Relative Errors Errors are often described relative to the size of the original quantity: $\mathrm{d}x = 5$ might be large or small compared to x.

In our previous example, the relative error in the length measurement x was

$$\frac{\mathrm{d}x}{x} = \frac{1/20}{10} = \frac{1}{200} = 0.5\%$$

while the resulting relative error in the volume V was

$$\frac{\mathrm{d}V}{V} = \frac{15}{1000} = \frac{3}{200} = 1.5\%$$

The error in the volume is therefore three times as significant as that in the length.

Homework

1. Find the value $x = b$ for which

$$\begin{cases} L_9(x) & \text{is a better approximation to } \sqrt{x} \text{ for } x < b \\ L_{16}(x) & \text{is a better approximation to } \sqrt{x} \text{ for } x > b \end{cases}$$

2. The Body Mass Index of a human is $B = \frac{m}{h^2}$, where m, h are the subject's mass and height.[2] Thus if $m = 77$ kg and $h = 1.74$ m, then $B = \frac{77}{1.74^2} = 25.43$.

   (a) Suppose that the mass is known exactly but that the height is only known up to some error $\mathrm{d}h$. Show that

   $$\mathrm{d}B = -\frac{2m}{h^3}\,\mathrm{d}h$$

   Compute the error in the BMI of our 77 kg subject if $\mathrm{d}h = 0.5$ cm.

   (b) Now suppose that the height is known exactly but that the mass is only known to be accurate to within 1%. Find the resulting error in the measurment of the BMI.

   (c) In multivariable calculus you will see that if B is viewed as a function of *both* h and m, then the total differential is

   $$\mathrm{d}B = \frac{1}{h^2}\left(\frac{\mathrm{d}}{\mathrm{d}m}m\right)\mathrm{d}m + m\left(\frac{\mathrm{d}}{\mathrm{d}h}\frac{1}{h^2}\right)\mathrm{d}h = \frac{1}{h^2}\,\mathrm{d}m - \frac{2m}{h^3}\,\mathrm{d}h$$

   Show that

   $$\frac{\mathrm{d}B}{B} = \frac{\mathrm{d}m}{m} - \frac{2\,\mathrm{d}h}{h}$$

   and that, in this situation, the maximum possible error in B is $\approx 1.57\%$. It would be appropriate to write $B = 25.43 \pm 0.40$.

[2] Measured in kilograms and meters respectively.