3.2 Product and Quotient Rules

While the derivatives of sums and difference follow the same pattern as the limit laws, the same is not true for products. For example, consider the following.

Example You might have thought that the derivative of a product should be the product of the derivatives: $\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) \stackrel{?}{=} f'(x)g'(x)$. Is this true? We can check with $x^2 = x \cdot x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x \cdot x) \stackrel{?}{=} 1 \cdot 1 = 1$$

However, by the power law, $\frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x$. Our guessed formula must be wrong.

Theorem (Product Rule)**.** *Suppose that f and g are differentiable. Then the product fg is differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Given all the symbols, you may find it easier to remember the Product Rule as follows:

$$(uv)' = u'v + uv'$$

where we've used the shorthand of $u = f(x)$ and $v = g(x)$. Note that the Product Rule gives the correct answer for differentiating $x^2 = x \cdot x$. Here $f(x) = g(x) = x$, and so $f'(x) = g'(x) = 1$, whence

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2 = 1 \cdot x + x \cdot 1 = 2x$$

It is still far easier to use the Power Law!

Example Find the derivative of x^2e^x with respect to x.

Here we may take $u = x^2$ and $v = e^x$, from which

$$u' = 2x \qquad v' = e^x$$

Putting it together gives

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2e^x = 2xe^x + x^2e^x = (2+x)xe^x$$

Again, the Product Rule should be a consequence of the limit definition of derivative. Indeed it is, for here is a proof:

Proof. Suppose that f and g are differentiable at x. Then

$$\begin{aligned}
\lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h} &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}g(x)+f(x+h)\frac{g(x+h)-g(x)}{h}\right] \\
&= f'(x)g(x)+f(x)g'(x)
\end{aligned}$$

Hence fg is differentiable at x with derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

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The proof is sneaky, for it requires you to insert and subtract off the same term in the second line. What matters is not the trickery, rather the fact that the Rule is seen to depend on the definition of derivative.

The Quotient Rule Unsurprisingly, there is a also a rule for differentiating one function divided by another.

Theorem (Quotient Rule)**.** *Suppose that f and g are differentiable. Then the quotient $\frac{f}{g}$ is differentiable whenever $g(x) \neq 0$, and*

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \qquad \text{(note the minus sign!)}$$

This can also be written

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{u}{v} = \frac{u'v - uv'}{v^2}$$

1. In this example, $f(x) = 1$ and $g(x) = x - 1$ both have very simple derivatives.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x-1} &= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}x}1\right)\cdot(x-1) - 1\cdot\left(\frac{\mathrm{d}}{\mathrm{d}x}(x-1)\right)}{(x-1)^2} = \frac{0\cdot(x-1) - 1\cdot 1}{(x-1)^2} = \frac{0-1}{(x-1)^2}\\ &= \frac{-1}{(x-1)^2}\end{aligned}$$

If this seems like too much algebra at once, break it up. For instance:

$$\begin{cases} u = 1 & \implies u' = 0 \\ v = x - 1 & \implies v' = 1 \end{cases}$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}x}\frac{u}{v} = \frac{u'v - uv'}{v^2} = \frac{0(x-1) - 1\cdot 1}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

2. Here is a more complicated example. There is lots of algebra involved, so take your time.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\sqrt{t}}{t^2-3} &= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{t}\right)(t^2-3) - \sqrt{t}\left(\frac{\mathrm{d}}{\mathrm{d}t}(t^2-3)\right)}{(t^2-3)^2}\\ &= \frac{\frac{1}{2}t^{-1/2}(t^2-3) - t^{1/2}\cdot 2t}{(t^2-3)^2}\\ &= \frac{-\frac{3}{2}t^{3/2} - \frac{3}{2}t^{-1/2}}{(t^2-3)^2} = \frac{-3(t^2+1)}{2\sqrt{t}(t^2-3)^2}\end{aligned}$$