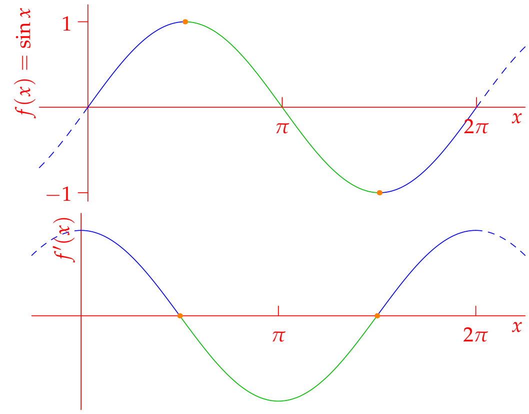
3.3 Derivatives of Trig functions

Differentiating polynomials is (relatively) easy. Unfortunately trigonometric functions require some more work:



Differentiating Sine The first thing to do is look at the picture. For reasons to be seen in a moment, we work in radians. We may visually differentiate the graph of $f(x) = \sin x$: think about where the curve is:

horizontal $\implies f'(x) = 0$
increasing $\implies f'(x) > 0$
decreasing $\implies f'(x) < 0$

$f'(x)$ certainly looks very like a cosine curve, but notice that we have no scale on the y-axis. We should suspect that $\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$, but we still need a *proof.*

Return to the definition of derivative. To differentiate $f(x) = \sin x$, we need to evaluate the limit as $h \to 0$ of the following difference quotient:

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{\sin(x+h) - \sin x}{h} \\ &= \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} && \text{(multiple-angle formula)} \\ &= \sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \end{aligned}$$

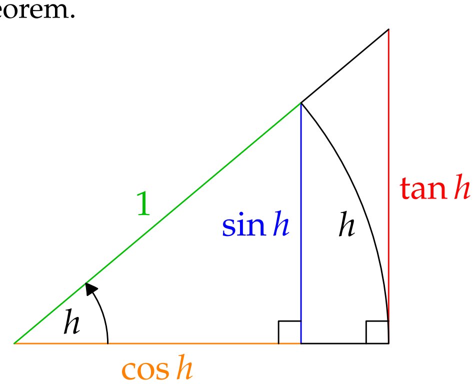
If our guess above is correct, then we need to prove the following:

Theorem. *If angles are measured in radians, then*

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \textit{and} \quad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0$$

It follows that $\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$.

Proof. Everything follows from the picture and the Squeeze Theorem.



Consider a segment of a circle of radius 1 and angle $h > 0$. By the definition of radians, the arc-length of the segment is also h. Drawing the triangles as shown, it should be clear that all lengths are as claimed.

It is immediate that $\sin h < h$, and so $\dfrac{\sin h}{h} < 1$.
Now recall that the area of the circular segment is

$$\frac{h}{2\pi} \cdot \text{area circle} = \frac{h}{2\pi} \cdot \pi = \frac{h}{2}$$

This must be less than the area of the large triangle, $\frac{1}{2} \tan h$. Therefore

$$\frac{h}{2} < \frac{1}{2} \tan h = \frac{\sin h}{2 \cos h} \implies \cos h < \frac{\sin h}{h}$$

We now have

$$\cos h < \frac{\sin h}{h} < 1$$

whenever $h > 0$, whence the Squeeze Theorem tells us that $\lim_{h \to 0^+} \frac{\sin h}{h} = 1$.
The limit as $h \to 0^-$ is the same, because $\frac{\sin h}{h}$ is an even function.
The second limit will be proved below. ■

Notice where we required radians in the proof: for finding the arc-length and area of the circular segment. If you repeat the exercise measuring angles in degrees, you obtain the horrible expressions:

$$\lim_{h \to 0} \frac{\sin h^\circ}{h^\circ} = \frac{\pi}{180} \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}x^\circ} \sin x^\circ = \frac{\pi}{180} \cos x^\circ$$

The moral of the story: always work in radians!

Other trigonometric functions These are easily dealt with via a similar argument to the above, or using the quotient rule.

Theorem. *The trigonometric functions have the following derivatives: all angles are measured in radians.*

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \tan x = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \cot x = -\csc^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x} \sec x = \sec x \tan x \qquad \frac{\mathrm{d}}{\mathrm{d}x} \csc x = -\csc x \cot x$$

Example proof. For $\tan x$ we use the quotient rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \tan x &= \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin x}{\cos x} = \frac{\frac{\mathrm{d}}{\mathrm{d}x} \sin x \cdot \cos x - \sin x \cdot \frac{\mathrm{d}}{\mathrm{d}x} \cos x}{\cos^2 x} \\ &= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \end{aligned}$$

■

Examples

1. Find the derivative of $f(x) = x^2 \tan x$.

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} x^2 \cdot \tan x + x^2 \cdot \frac{\mathrm{d}}{\mathrm{d}x} \tan x && \text{(product rule)} \\ &= 2x \tan x + x^2 \sec^2 x \end{aligned}$$

2. Compute $\frac{\mathrm{d}}{\mathrm{d}t} \sin 2t$. For this we start with the double angle formula.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \sin 2t &= \frac{\mathrm{d}}{\mathrm{d}t} 2 \sin t \cos t \\ &= 2 \left(\frac{\mathrm{d}}{\mathrm{d}t} \sin t \right) \cdot \cos t + 2 \sin t \cdot \frac{\mathrm{d}}{\mathrm{d}t} \cos t && \text{(product rule)} \\ &= 2\cos^2 t - 2\sin^2 t = 2\cos 2t \end{aligned}$$

3. Find the derivative of $s(t) = 3e^t \cos^2 t$. This problem requires a double application of the product rule. Think about the function as $s(t) = uv$, where $u = 3e^t$ and $v = \cos^2 t$. Then

$$u' = 3e^t$$
$$v' = \frac{\mathrm{d}}{\mathrm{d}t}(\cos t \cos t) = -\sin t \cos t + \cos t(-\sin t) = -2\sin t \cos t = -\sin 2t$$

where we used the product rule to differentiate v. Now use the product rule to combine these:

$$s'(t) = u'v + uv' = 3e^t \cos^2 t + 3e^t(-\sin 2t) = 3e^t(\cos^2 t - \sin 2t)$$

Limits of Trigonometric Functions The limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$ appears a lot. It can be used to simplify other related limits. For example suppose we want to compute

$$\lim_{x\to 0} \frac{\tan 2x}{\sin 3x}$$

The limit is the indeterminate form $\frac{0}{0}$
However, if we convert $\tan 2x = \frac{\sin 2x}{\cos 2x}$ and then insert multiples of x, we can appeal to the above limit:

$$\begin{aligned}\lim_{x\to 0} \frac{\tan 2x}{\sin 3x} &= \lim_{x\to 0} \sin 2x \cdot \frac{1}{\cos 2x} \cdot \frac{1}{\sin 3x} \\ &= \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{1}{\cos 2x} \cdot \frac{3x}{\sin 3x} \cdot \frac{2}{3} \\ &= 1 \cdot 1 \cdot 1 \cdot \frac{2}{3} = \frac{2}{3}\end{aligned}$$

In the second line, we inserted multiples of x in order to get expressions of the form $\frac{\sin(\text{lump})}{\text{lump}}$. The factor of $\frac{2}{3}$ is necessary to balance the extra x terms.

Homework

1. For what values of x does the graph of $y = 3e^x \sin x$ have a horiontal tangent line. Sketch the function. (*First draw $y = 3e^x$, then recall that $-1 \le \sin x \le 1 \ldots$*)

2. (a) Consider the picture on the right. Apply Pythagoras' Theorem to the red triangle to prove that

$$(\cos h - 1)^2 + \sin^2 h \le h^2$$

(b) Use this to prove that $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$.