3.4 The Chain Rule

Compound rates of change Suppose that Cora runs twice as fast as Bill who runs at three miles per hour. Suppose that $b(t)$ and $c(t)$ are the distances travelled by the protagonists at time t. Then we certainly have

$$\frac{\mathrm{d}b}{\mathrm{d}t} = 3$$

However, Cora's position will be changing *relative to Bill's:* she runs twice as fast, so we may also write

$$\frac{\mathrm{d}c}{\mathrm{d}b} = 2$$

If we now ask how rapidly Cora's position is changing *relative to* t, the answer is obvious: twice as fast as Bill means 6 miles per hour. The critical observation for this section is that this is the *product* of the two given rates of change:

$$6 = \frac{\mathrm{d}c}{\mathrm{d}t} = \frac{\mathrm{d}c}{\mathrm{d}b} \cdot \frac{\mathrm{d}b}{\mathrm{d}t} = 2 \cdot 3$$

Theorem (Chain Rule). *Suppose that f is differentiable at $g(a)$ and that g is differentiable at a. Then $f \circ g$ is differentiable at a and*[1]

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=a} (f \circ g)(x) = \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=a} f(g(x)) = f'\big(g(a)\big)g'(a)$$

Alternatively, if $u = g(x)$, then we may write this in Leibniz's notation,

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

In Leibniz's notation it appears that the expressions $\mathrm{d}u$ are being cancelled from top and bottom.

Example Find $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin 3x$

We write $y = f(g(x))$ where $u = g(x) = 3x$ and $f(u) = \sin u$. Hence

$$\frac{\mathrm{d}f}{\mathrm{d}u} = f'(u) = \cos u \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = g'(x) = 3$$

Therefore $f'(g(x)) = \cos 3x$ and so

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin 3x = f'(g(x))g'(x) = (\cos 3x) \cdot 3 = 3\cos 3x$$

More generally, $\frac{\mathrm{d}}{\mathrm{d}x} \sin kx = k \cos kx$ for any constant k.

[1] Most textbooks write $\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=a} (f(g(a)) = g'(a)f'\big(g(a)\big)$. There is no difference! We write it this way so that you can compare the two notations: $u = g(x) \implies \frac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$.

Sketch Proof. Suppose[2] that $g(x+h) \neq g(x)$ whenever $h \neq 0$. Then

$$\begin{aligned}
\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \cdot \frac{g(x+h)-g(x)}{g(x+h)-g(x)} \\
&= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \cdot \frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)} \\
&= g'(x) \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)}
\end{aligned}$$

Now let $t = g(x+h) - g(x)$, so that $h \to 0 \iff t \to 0$

$$\begin{aligned}
&= g'(x) \lim_{t\to 0} \frac{f(g(x)+t) - f(g(x))}{t} \\
&= g'(x) \cdot f'(g(x))
\end{aligned}$$

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As the proof hopefully makes clear, we differentiate f *then* evaluate f' at $g(x)$.

'Lumps': start with the outside function first When calculating examples using the chain rule, you may find it useful to to think about $g(x)$ as an abstract 'lump,' so that $f(g(x))$ is $f(\text{lump})$: concentrate on differentiating f first, then worry about the derivative of the lump. That is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(\text{lump}) = f'(\text{lump}) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\text{lump})$$

This is especially useful for examples where the chain rule is required more than once.

Example Differentiate $y = \sin(x^2)$. Treat x^2 as the 'lump,' and ignore it during the first step: you might want to write as follows:

Step 1 $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin(x^2) = \cos(\qquad) \cdot \dfrac{\mathrm{d}}{\mathrm{d}x}(\qquad)$ (since the derivative of sine is cosine)

Step 2 $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin(x^2) = \cos(x^2) \cdot \dfrac{\mathrm{d}}{\mathrm{d}x}(x^2)$ (substitute in the lump)

Step 3 $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin(x^2) = \cos(x^2) \cdot 2x$ (differentiate the lump)

Step 4 $\dfrac{\mathrm{d}}{\mathrm{d}x} \sin(x^2) = 2x\cos(x^2)$ (rearrange for the final answer)

Of course, you wouldn't write each step separately. On your paper it might ultimately look like the following,

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin(x^2) = \cos(x^2) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x\cos(x^2)$$

and the reader would never know you'd been thinking lumpy thoughts!

[2]This restriction is why the proof is a 'sketch.' Overcoming it is somewhat messy…

Harder Examples You should be able to write out complete arguments for the following either using the basic method (define f, g, etc.) or using lumps.

1.
$$\begin{aligned}\frac{d}{dx}(x^4+5x^2)^{1/3} &= \frac{1}{3}(x^4+5x^2)^{-2/3}\left(\frac{d}{dx}(x^4+5x^2)\right)\\ &= \frac{1}{3}(x^4+5x^2)^{-2/3}(4x^3+10x)\\ &= \frac{4x^3+10x}{3(x^4+5x^2)^{2/3}}\end{aligned}$$

2.
$$\begin{aligned}\frac{d}{dx}\cos(x+7x^{-1}) &= (-\sin(x+7x^{-1}))\left(\frac{d}{dx}(x+7x^{-1})\right)\\ &= (-\sin(x+7x^{-1}))(1-7x^{-2})\\ &= (7x^{-2}-1)\sin(x+7x^{-1})\end{aligned}$$

3. This example uses the chain rule twice:

$$\begin{aligned}\frac{d}{dx}\tan(\sin(x^3)) &= \sec^2(\sin(x^3))\cdot\left(\frac{d}{dx}\sin(x^3)\right) && \text{(first 'lump' is } \sin(x^3)\text{)}\\ &= \sec^2(\sin(x^3))\cdot\cos(x^3)\cdot\left(\frac{d}{dx}x^3\right) && \text{(second 'lump' is } x^3\text{)}\\ &= 3x^2\cos(x^3)\sec^2(\sin(x^3)) && \text{(simplify and rearrange)}\end{aligned}$$

Homework

1. Differentiate $\sin(\sin(\sin(\sin x)))$.
2. Suppose that the function $f(x)$ satisfies the equation $[f(x)]^3 + x\sin(f(x)) = x^3 + 8$.
   (a) Show first that $f(0) = 2$.
   (b) Now prove that $f'(0) = -\frac{1}{12}\sin 2$.
   (c) Try to compute $f''(0)$ (*the answer's very ugly…*)