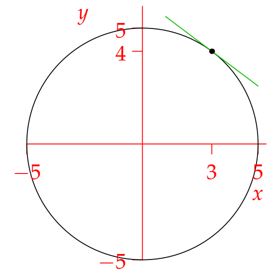
3.5 Implicit Differentiation

An *implicit curve* is a the set of points (x, y) satisfying an equation of the form $f(x, y) = 0$. For example, the equation $x^2 + y^2 = 25$ describes a circle of radius 5 centered at the origin. Suppose that we wanted to find the slope of this curve at the point $(3, 4)$. If the curve had an *explicit* equation $y = f(x)$, then we would simply be looking to compute $f'(3)$. Indeed this is one of the ways to answer the question.



Method 1 First solve for y in terms of x. We obtain

$$y = \sqrt{25 - x^2}$$

where we are taking the positive square root since our point of interest $(3, 4)$ has a positive y-co-ordinate. Now compute the derivative using the chain rule.

$$\frac{dy}{dx} = \frac{d}{dx}(25 - x^2)^{-1/2} = \frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{25 - x^2}}$$

Evaluating this at $x = 3$ gives the required slope:

$$\left.\frac{dy}{dx}\right|_{x=3} = \frac{-3}{\sqrt{25 - 3^2}} = -\frac{3}{4}$$

Method 2 Differentiate both sides of the implicit equation with respect to x, treating y as a function of x. We do not need to know what $y(x)$ is, we simply treat it as a 'lump' for the purposes of the chain rule. In this case we obtain

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}1 \implies 2x + 2y\frac{dy}{dx} = 0 \qquad \left(\frac{d}{dx}(y(x))^2 = 2y(x)\frac{d}{dx}y(x)\right)$$

$$\implies \frac{dy}{dx} = -\frac{x}{y}$$

$$\implies \left.\frac{dy}{dx}\right|_{(x,y)=(3,4)} = -\frac{3}{4}$$

The second method is known as *implicit differentiation.* Hopefully you agree that it is the easier method! There is no difficulty with square roots, and the application of the chain rule, while abstract, is also simpler. One extra challenge with implicit differentiation is that the result is an expression for $\frac{dy}{dx}$ as a function of *both* x and y. You therefore need to know both co-ordinates of a point before you differentiate. *Explicit* differentiation only requires knowledge of x. More importantly, the above expression $\frac{dy}{dx} = -\frac{x}{y}$ *only* makes sense for those points (x, y) lying *on the circle* and where the circle has a finite slope. For instance, $\left.\frac{dy}{dx}\right|_{(x,y)=(10,1)}$ and $\left.\frac{dy}{dx}\right|_{(x,y)=(5,0)}$ are both meaningless.

How to implicitly differentiate

1. Differentiate both sides of an equation with respect to x.
2. Differentiate any expressions involving x normally.
3. Use the chain rule to differentiate any expressions involving y: e.g. $\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$.
4. Rearrange to isolate $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

In most cases, when given an implicit curve, you have no option but to use implicit differentiation. *Method 1* relies on your being able to solve explicitly for y, which cannot be done in general.

Examples

1. Find the equation of the tangent line to the curve $x^2 + y^3 - 2xy = 1$ at the point $(2, 1)$.
   In order to perform an implicit differentiation, we will need to find $\frac{\mathrm{d}}{\mathrm{d}x}y^3 = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$, and use the product rule to compute

   $$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = \left(\frac{\mathrm{d}}{\mathrm{d}x}x\right) \cdot y + x \cdot \frac{\mathrm{d}}{\mathrm{d}x}y = y + x\frac{\mathrm{d}y}{\mathrm{d}x}$$

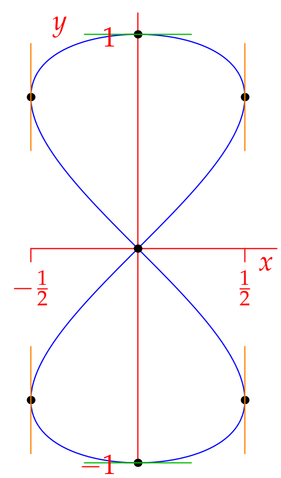
   Putting this all together, and differentiating both sides with respect to x, we obtain

   $$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^3 - 2xy) = \frac{\mathrm{d}}{\mathrm{d}x}1 \implies 2x + 3y^3\frac{\mathrm{d}y}{\mathrm{d}x} - 2y - 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
   $$\implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - 2y}{2x - 3y^2}$$

   The tangent line at $(2, 1)$ therefore has gradient $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x,y)=(2,1)} = \frac{2}{1} = 2$ and therefore has equation

   $$y - 1 = 2(x - 2) \iff y = 2x - 3$$

2. Find the points on the implicit curve $x^2 = y^2(1 - y^2)$ at which the gradient is horizontal, then vertical. What is happening at $(0, 0)$?

   Implicitly differentiating we obtain

   $$2x = 2y\frac{\mathrm{d}y}{\mathrm{d}x}(1 - y^2) - 2y^3\frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y(1 - 2y^2)}$$

   Ignoring the origin which gives the meaningless $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{0}{0}$, we get

   Horizontal $\iff x = 0 \iff (x, y) = (0, \pm 1)$

   Vertical $\iff 1 - 2y^2 = 0 \iff y = \pm\frac{1}{\sqrt{2}} \iff (x, y) = \left(\pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$

   Finally we deal with $(0, 0)$. Squaring we obtain

   $$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{x^2}{y^2(1 - 2y^2)^2} = \frac{y^2(1 - y^2)}{y^2(1 - 2y^2)^2} = \frac{1 - y^2}{(1 - 2y^2)^2} \xrightarrow[y \to 0]{} 1$$

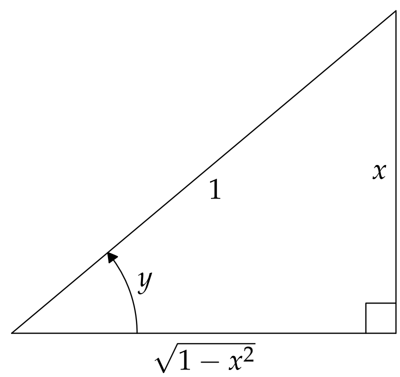
   which yields gradients of ± 1 at $(0, 0)$

Differentiating Inverse Functions

Implicit differentiation can be used to compute the derivatives of inverse functions. For example, here is how to find the derivative of $y = \sin^{-1} x = \arcsin x$.

- $y = \sin^{-1} x$ is equivalent to $x = \sin y$.
- Implicitly differentiate:
$$\frac{\mathrm{d}}{\mathrm{d}x} x = \frac{\mathrm{d}}{\mathrm{d}x} \sin y \implies 1 = \cos y \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$
- Rearrange $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\cos y} = \dfrac{1}{\cos(\sin^{-1} x)}$.

The last expression is very ugly. It can be simplified by thinking about an abstract triangle: since the expression contains $\cos y$ we draw a right-triangle that has y as an *angle.* By assumption, $\sin y = x$, so we label the sides of the triangle in such a manner that this relation holds: namely $\sin y = x = \frac{\text{opposite}}{\text{hypotenuse}}$. Pythagoras' Theorem gives the length $\sqrt{1 - x^2}$ of the adjacent. From the triangle, it is immediate that $\cos y = \frac{1}{\sqrt{1-x^2}}$, and so



$$\frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

More generally, we can use implicit differentiation to prove the following:

Theorem. *If f is an invertible function, then* $\dfrac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \dfrac{1}{f'(f^{-1}(x))}$

The Theorem can be used to find the derivative of an inverse function at a particular point, *without* having to compute the inverse function explicitly.

Example Suppose $y = f(x) = x^5 + 3x^2 - 29x$. Find $\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x)$ at $x = -25$. It should be clear that $-25 = f(1)$. Applying the Theorem, we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=-25} f^{-1}(x) = \frac{1}{f'(f^{-1}(-25))} = \frac{1}{f'(1)}$$

However $f'(x) = 5x^4 + 6x - 29 \implies f'(1) = 5 + 6 - 29 = -18$. It follows that

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=-25} f^{-1}(x) = -\frac{1}{18}$$

Homework

1. The Tschirnhausen cubic is the curve $x^3 = 9a\left(x^2 - 3y^2\right)$, where a is a constant. Prove that the tangent line to the cubic is horizontal if $x = 6a$. Can you explain what the curve is doing at $x = 0$, and why it is not strictly correct to say that the tangent is horizontal?
2. Prove the Theorem on on the derivative of an inverse function.