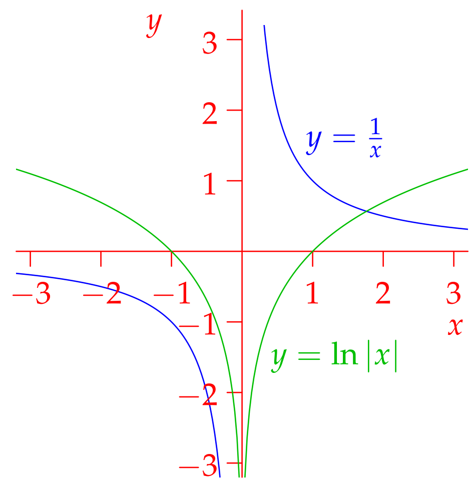
3.6 Derivatives of Logarithmic Functions

Recall how to differentiate inverse functions using implicit differentiation. Since the natural logarithm is the inverse function of the natural exponential, we have

$$y = \ln x \iff e^y = x \implies e^y \frac{\mathrm{d}y}{\mathrm{d}x} = 1 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^y} = \frac{1}{x}$$



We have therefore proved the first part of the following Theorem: the remainder follow immediately using the log laws and chain rule.

Theorem.

- *If* $x > 0$ *then* $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$
- *If* $x \neq 0$, *then* $\frac{\mathrm{d}}{\mathrm{d}x} \ln |x| = \frac{1}{x}$
- $\frac{\mathrm{d}}{\mathrm{d}x} \log_a x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\ln x}{\ln a} = \frac{1}{x \ln a}$
- $\frac{\mathrm{d}}{\mathrm{d}x} a^x = \frac{\mathrm{d}}{\mathrm{d}x} e^{x \ln a} = (\ln a) e^{x \ln a} = (\ln a) a^x$

The last two parts of the Theorem illustrate why calculus always uses the *natural* logarithm and exponential. Any other base causes an extra factor of $\ln a$ to appear in the derivative. Recall that $\ln e = 1$, so that this factor never appears for the natural functions.

Example We can combine these rules with the chain rule. For example:

$$\frac{\mathrm{d}}{\mathrm{d}x} \log_4(x^2+7) = \frac{1}{(x^2+7)(\ln 4)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x^2+7) = \frac{2x}{(x^2+7)(\ln 4)}$$

Logarithmic Differentiation

This is a powerful technique, allowing us to use the log laws to simplify an expression *before* differentiating. For example, suppose that you wanted to differentiate

$$g(x) = \ln \frac{3x^2+1}{\sqrt{1+x^2}}$$

Blindly applying the chain rule, we would treat g as ln(lump), then be stuck with a hideously unpleasant quotient rule calculation:

$$\begin{aligned} g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \ln \frac{3x^2+1}{\sqrt{1+x^2}} = \frac{1}{\frac{3x^2+1}{\sqrt{1+x^2}}} \frac{\mathrm{d}}{\mathrm{d}x} \frac{3x^2+1}{\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}}{3x^2+1} \frac{\mathrm{d}}{\mathrm{d}x} \frac{3x^2+1}{\sqrt{1+x^2}} \\ &= \frac{\sqrt{1+x^2}}{3x^2+1} \cdot \frac{6x\sqrt{1+x^2} - (3x^2+1) \cdot x(1+x^2)^{-1/2}}{1+x^2} \\ &= \frac{1}{3x^2+1} \cdot \frac{6x(1+x^2) - (3x^2+1) \cdot x}{1+x^2} = \frac{x(3x^2+5)}{(3x^2+1)(1+x^2)} \end{aligned}$$

Thankfully there is a better way: apply the logarithm laws first,

$$g(x) = \ln \frac{3x^2+1}{\sqrt{1+x^2}} = \ln(3x^2+1) - \frac{1}{2}\ln(1+x^2)$$

and *then* differentiate:

$$\begin{aligned} g'(x) &= \frac{1}{3x^2+1}\frac{\mathrm{d}}{\mathrm{d}x}(3x^2+1) - \frac{1}{2(1+x^2)}\frac{\mathrm{d}}{\mathrm{d}x}(1+x^2) \\ &= \frac{6x}{3x^2+1} - \frac{x}{1+x^2} \end{aligned}$$

A little algebra shows that we have the same solution, in a *much* simpler way.

Logarithmic differentiation is so useful, that it is most often applied to expressions which do not contain any logarithms at all. Suppose instead that we had wanted to differentiate

$$f(x) = \frac{3x^2+1}{\sqrt{1+x^2}}$$

Then $g(x) = \ln f(x)$ is easy to differentiate and, since

$$g'(x) = \frac{f'(x)}{f(x)} \implies f'(x) = f(x)g'(x)$$

we can immediately write the derivative:

$$f'(x) = \frac{3x^2+1}{\sqrt{1+x^2}}\left[\frac{6x}{3x^2+1} - \frac{x}{1+x^2}\right]$$

Logarithmic Differentiation in General To differentiate $y = f(x)$ using logarithmic differentiation.

1. Take natural logs of both sides: $\ln y = \ln f(x)$
2. (Implicitly) Differentiate with respect to x: the left hand side becomes $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$
3. Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

This is especially useful if the form of $f(x)$ is something that simplifies nicely under the logarithm laws: for example products, quotients and powers.

Examples

1. Let $y = \dfrac{(2x+1)^{1/3}(1+x^2)}{(2+x^4)^{2/5}}$. Then

$$\begin{aligned} \ln y &= \frac{1}{3}\ln(2x+1) + \ln(1+x^2) - \frac{2}{5}\ln(2+x^4) \\ \implies \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2}{3(2x+1)} + \frac{2x}{1+x^2} - \frac{8x^3}{2+x^4} \\ \implies \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(2x+1)^{1/3}(1+x^2)}{(2+x^4)^{2/5}}\left[\frac{2}{3(2x+1)} + \frac{2x}{1+x^2} - \frac{8x^3}{2+x^4}\right] \end{aligned}$$

2. Let $y = (\sin x)^{\sqrt{x}}$. Then $\ln y = \sqrt{x}\ln(\sin x)$, from which

$$\begin{aligned}\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}x^{-1/2}\ln(\sin x) + \sqrt{x}\cdot\frac{\cos x}{\sin x} && \text{(product rule)}\\ &= \frac{1}{2\sqrt{x}}\left[\ln(\sin x) + 2\cot x\right]\\ \implies \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y}{2\sqrt{x}}\left[\ln(\sin x) + 2\cot x\right] = \frac{(\sin x)^{\sqrt{x}}}{2\sqrt{x}}\left[\ln(\sin x) + 2\cot x\right]\end{aligned}$$

***e* as a limit**

We have already seen the suggestion that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ when we discussed compound interest. Here is a proof, using the definition of derivative. Let $f(x) = \ln x$, then

$$\frac{1}{x} = \frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0^+}\frac{\ln(x+h)-\ln x}{h} = \lim_{h\to 0^+}\ln\left(\frac{x+h}{x}\right)^{1/h}$$

Now write $n = \frac{1}{h}$ and $y = \frac{1}{x}$ to obtain

$$y = \lim_{n\to\infty}\ln\left(1+\frac{y}{n}\right)^n = \ln\lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n \implies e^y = \lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n$$

This is usually written as

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \qquad \text{or} \qquad \lim_{x\to 0}(1+x)^{1/x} = e$$

Homework

1. If $x^y = y^x$, use implicit and logarithmic differentiation to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
2. Suppose that a is constant and the functions f and g are related by

   $$f(x) = a^{g(x)}$$

   Prove that $f'(x) = (\ln a)g'(x)a^{g(x)}$.
3. Similarly to the previous question, compute

   $$\frac{\mathrm{d}}{\mathrm{d}x}(g(x))^{h(x)}$$