3.8 Exponential Growth and Decay

Recall the definition of the natural exponential function: it is *the* exponential function whose derivative is equal to itself. The exponential function is in fact more powerful than this: it can be used to describe any process where the rate of change of the output is proportional to[1] the output itself.

Example: population growth We might expect that, as a population increases, the number of babies born will also increase. A simple model might say that if one doubles the population, the number of babies born each year would also double: this is *direct proportion.*[2] The rate of change of population is a constant multiple of the population itself. If we let $P(t)$ represent the population at time t, then we can write this mathematically:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \tag{$*$}$$

where $k > 0$ is some positive constant. Here $\frac{\mathrm{d}P}{\mathrm{d}t}$ is the rate of increase in the population (number of babies born per year—for simplicity we assume no-one dies). The *differential equation* $(*)$ has a special name.

Definition. *The* natural growth equation *is the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky$$

where $k > 0$ is constant. k is known variously as the growth constant, *or* natural growth rate, *or* rate of natural increase.
If $k < 0$, the equation is known as the natural decay equation. *The names for k change similarly.*

Theorem. *If $y(x)$ satisfies the natural growth/decay equation, then*

$$y(x) = y(0)e^{kt}$$

where $y(0)$ is the value of y when $x = 0$.

Exponential functions are therefore precisely the solutions of the natural growth/decay equations.

There are many situations in the sciences when the natural growth equation applies. Here are a few examples.

Population of bacteria Suppose that a population of bacteria grows naturally. Suppose that the initial population is 100 bacteria, and, after 1 hour, the population has grown to 120. How large will the population be after 2 hours? How long will it take for the population to reach 1,000,000 bacteria?

Suppose that the population at time t hours is given by the function $P(t)$. We are told the following:

[1] A constant multiple of.

[2] This is a very simple model. There is good evidence to suggest that that people typically have *fewer* babies per person when population is high.

P grows naturally Therefore $\frac{dP}{dt} = kP$ for some positive constant k.

Initial population $P(0) = 100$

Population at 1 hour $P(1) = 120$

By the Theorem, we know that

$$P(t) = P(0)e^{kt} = 100e^{kt}$$

since we know the initial population. We need only evaluate this expression at $t = 1$ to find the constant k.

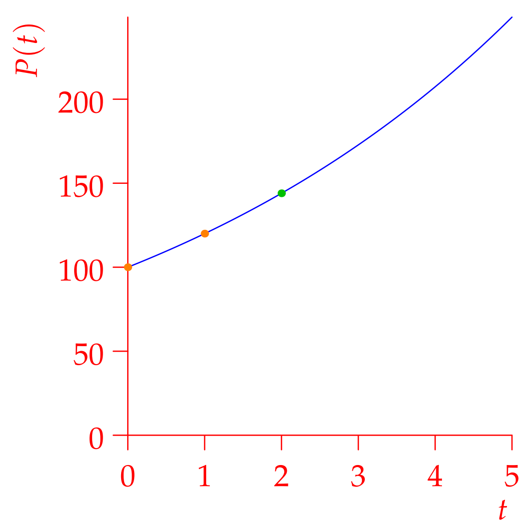
$$120 = P(1) = 100e^{k} \implies e^{k} = \frac{6}{5} = 1.2 \implies k = \ln\frac{6}{5}$$

In this case it is not so useful to find k directly, e^k is sufficient to obtain a general formula:

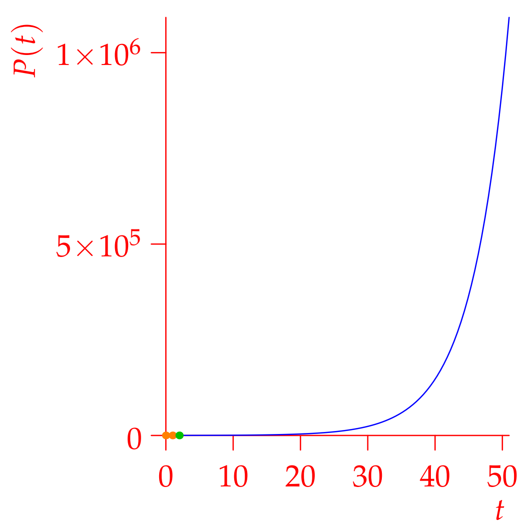
$$P(t) = 100e^{kt} = 100(e^k)^t = 100\left(\frac{6}{5}\right)^t$$

After 2 hours we have $P(2) = 100\left(\frac{6}{5}\right)^2 = 100 \cdot \frac{36}{25} = 144$ bacteria.
To find how long it takes to reach 1,000,000 bacteria, we solve[3]

$$\begin{aligned} 100\left(\frac{6}{5}\right)^t = 1000000 &\implies \left(\frac{6}{5}\right)^t = 10000 \\ &\implies t\ln\frac{6}{5} = \ln 10000 \\ &\implies t = \frac{\ln 10000}{\ln 6 - \ln 5} \approx 50.5 \text{ hours} \end{aligned}$$



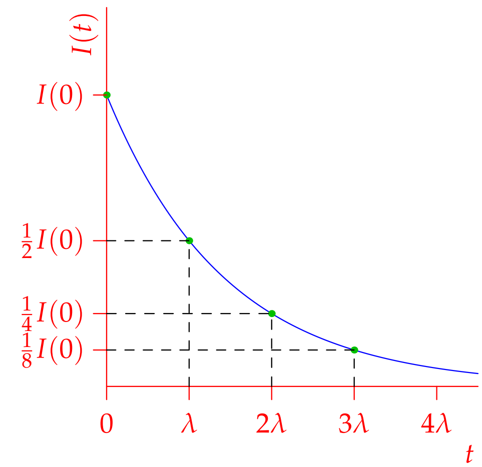
Initial points in orange



Same graph at greater scale

[3]Because of the powers of 10, it is actually simpler to use $\log_{10}$ for evaluating on your calculator: $t = \frac{\log_{10} 10000}{\log_{10} 1.2} = \frac{4}{\log_{10} 1.2}$. In calculus, natural log is safer!

Half-life Radioactive materials emit radiation (energy) and, as they do so, they transform (decay) into other materials. The *half-life*, usually denoted λ, of a material is the time taken for the intensity of the radiation to drop to half its initial value. It follows that the intensity of the radiation satisfies an equation



$$I(t) = I(0) \cdot 2^{-t/\lambda}$$

Thus at time λ, the intensity is $\frac{1}{2}$ the original, at time 2λ the intensity is $\frac{1}{4}$ the original, etc. As a differential equation, we have

$$\frac{\mathrm{d}I}{\mathrm{d}t} = I(0) \cdot \frac{-1}{\lambda} \cdot (\ln 2) \cdot 2^{-t/\lambda} = \frac{-\ln 2}{\lambda} I$$

which is the natural decay equation with $k = -\frac{\ln 2}{\lambda}$. We should check: by the Theorem, the solution is

$$I(t) = I(0)e^{kt} = I(0)e^{-\frac{t \ln 2}{\lambda}}$$

which might look different. However, by the exponential laws,

$$e^{-\frac{t \ln 2}{\lambda}} = (e^{\ln 2})^{-\frac{t}{\lambda}} = 2^{-t/\lambda}$$

so we get the same equation. The same equations are satisfied by the *mass* of a radioactive substance. For example:

Question: A spy disarms a 60 year old nuclear warhead, originally containing 1kg of pure plutonium 239. Pu^{239} has a half-life of 24,200 years: what mass of plutonium remains?

Solution: We have $\lambda = 24200$ years and initial mass $m(0) = 1$ kg. It follows that

$$m(t) = m(0)2^{-t/24200} = 2^{-t/24200}$$

After $t = 60$ years, there remains

$$m(60) = 2^{-60/24200} \approx 0.99828 \text{ kg } = 998.28 \text{ grams of plutonium.}$$

The missing mass will (mostly) be uranium 235 and other decay products.

Newton's Law of Cooling Newton's law of cooling is the basic assumption that the rate of change of temperature of a body should be proportional to the *difference* between its temperature and that of the surroundings.

If we let $T(t)$ be the temperature of the body at time t, and T_s the temperature of the surroundings, this may be written as a differential equation:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_s)$$

where $k > 0$ is constant. Note the sign of the constant: if the body is hotter than the surroundings ($T > T_s$), then we expect the body to cool down ($\frac{\mathrm{d}T}{\mathrm{d}t} < 0$).

The differential equation can be solved by substitution: if we let $y(t) = T(t) - T_s$, then[4]

$$\frac{dy}{dt} = -ky$$

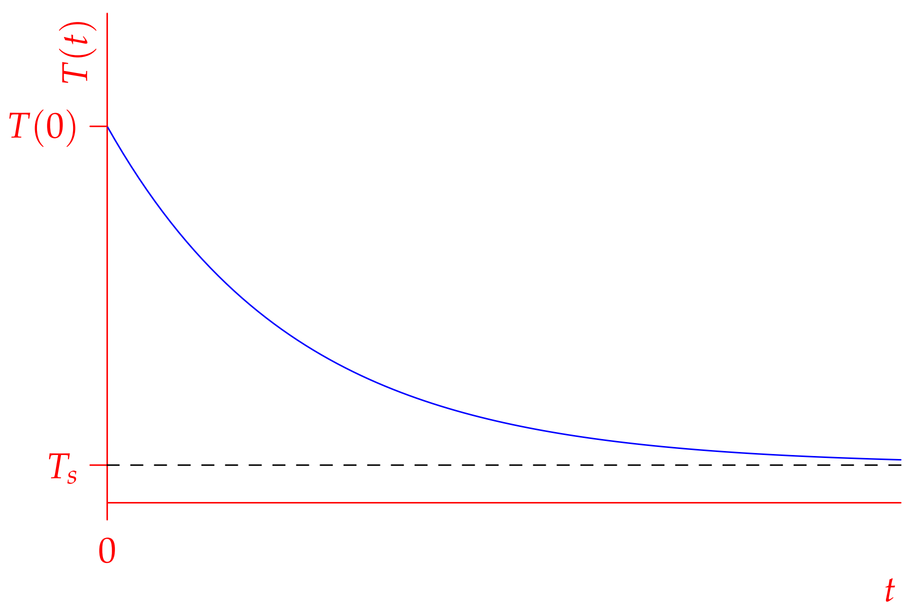
This is the natural decay equation, with solution

$$y(t) = y(0)e^{-kt}$$

Substituting back for $y(t) = T(t) - T_s$ yields

$$T(t) = T_s + (T(0) - T_s)e^{-kt}$$

With a little thought, you should be convinced that the graph of the solution is as follows:



Of course a scientist would then test Newton's Law by experimenting, say with a cup of coffee...

Example Suppose that a physicist wants to test Newton's Law of cooling. They take a cup of hot coffee (200° F) outside on a warm day (80° F) and test the temperature. They record the following data.

Time (mins)	0	4	8	12
Temp (° F)	200	140	110	100

1. Use the data values at $t = 0$ and 4 minutes to find an expression for $T(t)$, assuming Newton's Law of cooling holds.

2. Are the observations at 8 and 12 minutes consistent with the Law?

[4]Since T_s is constant, $\frac{dy}{dt} = \frac{dT}{dt}$.

Assuming that Newton's Law holds, and applying the fact that $T_s = 80$ and $T(0) = 200$, we must have

$$T(t) = 80 + (200 - 80)e^{-kt} = 80 + 120e^{-kt}$$

Now evaluate at $t = 4$ minutes. We must have

$$140 = 80 + 120e^{-4k} \implies e^{-4k} = \frac{140 - 80}{120} = \frac{1}{2}$$

There is no need to compute k explicitly[5] since we can use e^{-4k} in the formula:

$$e^{-kt} = (e^{-4k})^{t/4} = \left(\frac{1}{2}\right)^{t/4}$$

We therefore have our model, assuming that Newton's Law holds:

$$T(t) = 80 + 120\left(\frac{1}{2}\right)^{t/4}$$

Comparing this with the observed temperatures in the table, we see that we should have

$$T(8) = 80 + 120\left(\frac{1}{2}\right)^{8/4} = 80 + 30 = 110^\circ\text{ F}$$

which is exactly as observed, and

$$T(12) = 80 + 120\left(\frac{1}{2}\right)^{12/4} = 80 + 15 = 95^\circ\text{ F}$$

which is below the observation. It might be that the last obervation was made a little late, or that the sun came out towards the end of the observations. Or indeed the model could be wrong: further experimentation is needed!

[5] It is $\frac{1}{4}\ln 2$.