3.9 Related Rates

If two quantites are related, then the rates of change of those quantities will also satisfy some relationship. This section is essentially a collection of word problems that should generally be attacked using the following steps (not all steps will necessarily apply to all problems)

1. Draw a picture and assign variables.
2. State every piece of information in the question in terms of the variables and identify what the question wants you to compute.
3. Differentiate something (usually requires the chain rule or implicit differentiation) to obtain a relationship between rates of change.
4. Substitute the information from part (b).
5. Check units and make sure you've answered the question.

Examples

1. Here is a very simple example, where no picture or interpretation is required.

   Suppose that the functions f and g are related by

   $$g(x) = (4f(x) + 20x)^2$$

   Suppose that $f(3) = 1$ and $f'(3) = -7$, find the rate of change of g at $x = 3$.

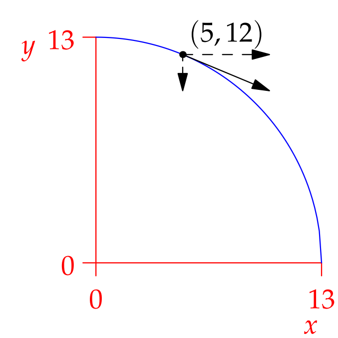
   We simply differentiate the relationship between f and g using the chain rule:

   $$g'(x) = 2(4f(x) + 20x)(4f'(x) + 20)$$

   Therefore

   $$g'(3) = 2(4f(3) + 20)(4f'(3) + 20) = 2(4 + 20)(-28 + 20) = -384$$

2. A goat is tethered to a rope of length 13 m. At a given time, the goat is at the point $(5, 12)$ and its x-co-ordinate is increasing at 1 m/s. If the rope remains tight, how rapidly is its y-co-ordinate changing?

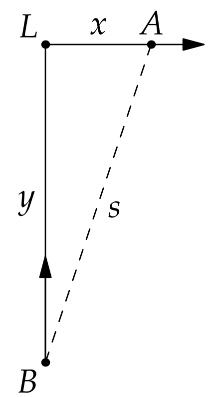
   Let the goat's co-ordinates be $(x(t), y(t))$ at time t.
   The goat moves in a circle of radius 13, hence $x^2 + y^2 = 13^2$.
   At the time of interest, we are given that $x = 5, y = 12, \frac{dx}{dt} = 1$.
   We want to calculate $\frac{dy}{dt}$.
   Now differentiate with respect to t:

   $$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}13^2 \implies 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
   $$\implies \frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

   The y-co-ordinate is therefore changing at a rate of

   $$\frac{dy}{dt} = -\frac{5}{12} \cdot 1 = -\frac{5}{12} \text{ m/s}$$

3. A ship is moving East away from a lighthouse at 9 mph, while another moves North towards the same lighthouse at 12 mph. How rapidly is the distance between the ships changing when the ships are 1 and 3 miles, respectively, from the lighthouse?

   Draw a picture and choose variables. Let the first ship be ship A and the second ship B. Let $x(t)$ denote the distance of ship A East of the lighthouse L. Let $y(t)$ be the distance of ship B South of the lighthouse. $s(t)$ denotes the distance between the ships.
   We know the following:

   $$s^2 = x^2 + y^2 \qquad \text{(by Pythagoras')}$$

   $$x = 1, \quad y = 3, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = 9, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -12 \qquad \text{(at the time of interest)}$$

   Note that $\frac{\mathrm{d}y}{\mathrm{d}t} < 0$ because the distance from B to L is *decreasing.*
   We want to compute $\frac{\mathrm{d}s}{\mathrm{d}t}$.
   Differentiate the Pythagorean relationship to obtain

   $$2s\frac{\mathrm{d}s}{\mathrm{d}t} = 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} \implies \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{s}\left[x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}\right]$$

   At the time of interest, $s^2 = x^2 + y^2 = 1^2 + 3^2 = 10 \implies s = \sqrt{10}$ miles. Therefore

   $$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{\sqrt{10}}[1 \cdot 9 + 3 \cdot (-12)] = -\frac{25}{\sqrt{10}} \text{ mph}$$

4. The ideal gas law for a particular quantity of gas says that $PV = T$, where P is pressure (Pa), V is volume (m^3), and T is temperature (K)
   At a given time, suppose that the gas is being heated at a rate of 3 K/s, and the volume reduced at $\frac{1}{100}$ m^3/s. If, at this time, the volume is 2 m^3 and the pressure is 10000 Pa, find the rate of change of the pressure.

   This time we have all the necessary variables. The question states that

   $$PV = T, \qquad P = 10000, \qquad V = 2, \qquad \frac{\mathrm{d}T}{\mathrm{d}t} = 3, \qquad \frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{1}{100}$$

   Note that $\frac{\mathrm{d}V}{\mathrm{d}t} < 0$ because the volume is being reduced. Now differentiate the ideal gas law, using the product rule to deal with the left hand side:

   $$\frac{\mathrm{d}P}{\mathrm{d}t} \cdot V + P \cdot \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}T}{\mathrm{d}t} \implies 2\frac{\mathrm{d}P}{\mathrm{d}t} + 10000 \cdot \frac{-1}{100} = 3$$
   $$\implies \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{103}{2} \text{ Pa/s}$$

More example problems

1. Suppose that a forest fire is spreading in a circle. with radius increasing at 5ft/min. When the radius is 200 ft, how rapidly is the burn area increasing.

2. An observer stands 200 m from the launch site of a hot-air baloon. The balloon rises vertically at a constant rate of 5 m/s. How rapidly is the angle of elevation of the balloon increasing 30 s after the launch?

3. A swimming pool is 50m long and 10m wide, with a planar bottom, 6m and 1m deep at opposite ends. The pool is filled at a rate of $2\text{m}^3/\text{min}$. How fast is the water level rising when the depth is 3m?

4. A spherical balloon is being inflated. At time t seconds the volume is $V(t) = \frac{4\pi}{3}t\ \text{cm}^3$. Find the rate of change of the surface area of the balloon.

Answers:

1. $2000\pi\ \text{ft}^2/\text{min}$.

2. $\frac{2}{125}$ rad/s.

3. $\frac{1}{150}$ m/min.

4. $\frac{8\pi}{3}t^{-1/3}\ \text{cm}^2/\text{s}$