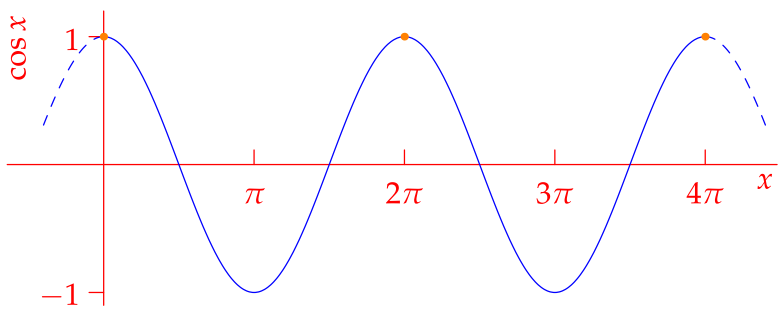
4 Applications of Differentiation

4.1 Maximum and Minimum Values

Many real-life problems can be rephrased in terms of maximizing or minimizing the value of a function. For example, 'How do we make the most profit?' or 'How can we save energy?' (minimize waste, or maximize efficiency). Calculus has a role to play in addressing these questions. First we need to formalize what we mean.

Definition. *$f(c)$ is the* absolute maximum *value of f if $f(c) \geq f(x)$ for all x.*
$f(c)$ is the absolute minimum *value of f if $f(c) \leq f(x)$ for all x.*
$f(c)$ is a local maximum *if $f(c) \geq f(x)$ for all x near c.*
$f(c)$ is a local minimum *if $f(c) \leq f(x)$ for all x near c.*

- If $f(c)$ is the absolute maximum value of f then we would also say that the point $(c, f(c))$ is, say, an *absolute maximum point* of f.
  The distinction between *the* and *an* is important: for instance $f(0) = 1$ is *the* absolute maximum value of $f(x) = \sin x$, but we could also write $f(2\pi) = 1$, or $f(4\pi) = 1$.



One absolute max *value*, many *points*

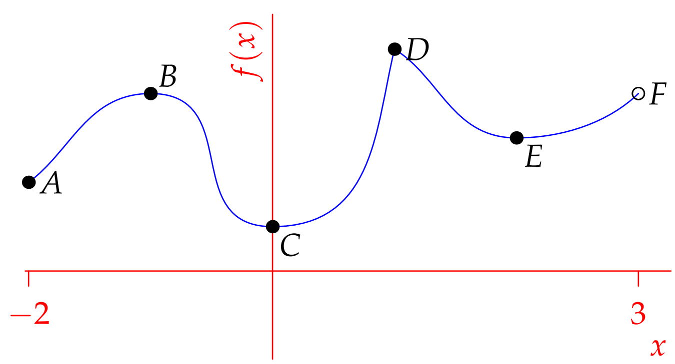
- The strict definition of 'near' when talking about local extrema is difficult and requires a discussion beyond the level of this course.

- Absolute maxima are also local maxima, etc.

- Horizontal lines will have absolute maximum and minimim values equal: for example $f(x) = 1$ has absolute maximum and minimum values of 1, at *all* values of x!

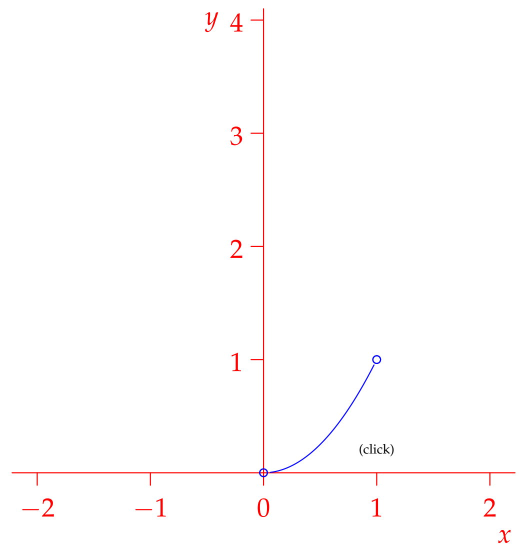
Example The function has domain $[-2, 3)$.
The types of each point are listed.

Point	Type
A	Local Minimum
B	Local Maximum
C	Local + Absolute Minimum
D	Local + Absolute Maximum
E	Local Minimum
F	n/a: not in graph of f



Domains The domain of a function is critical to the location and values of maxima and minima. For example, consider $f(x) = x^2$ where we let the domain be various intervals.

Domain	Maxima	Minima
$(0, 1)$	None	None
$(-1, 1)$	None	$(0, 0)$
$[-1, 1]$	$(-1, 1)$, $(1, 1)$	$(0, 0)$
$[-1, 2]$	$(-1, 1)$, $(2, 4)$	$(0, 0)$
$\mathbb{R}$	None	$(0, 0)$



Critical Points

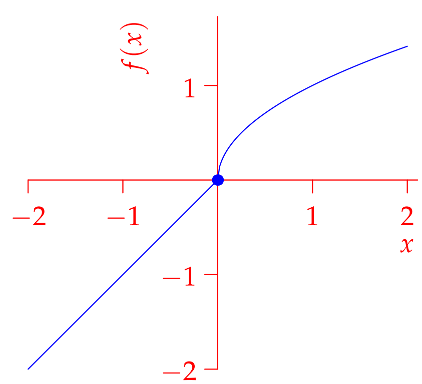
Definition. *Let f be a fucntion. We say that $x = c$ is a* critical value *of f if the derivative $f'(c)$ is either zero or undefined. We call $(c, f(c))$ a* critical point *of f.*

Recall the picture on the previous page: local maxima and minima which are not endpoints of a curve appear to be critical points. Indeed this is a theorem:

Theorem (Fermat)**.** *Suppose that f is defined on an interval I and that c is* not *an endpoint of I. If $f(c)$ is a local maximum or minimum value of f, then c is a critical value of f.*

The converse however is false.

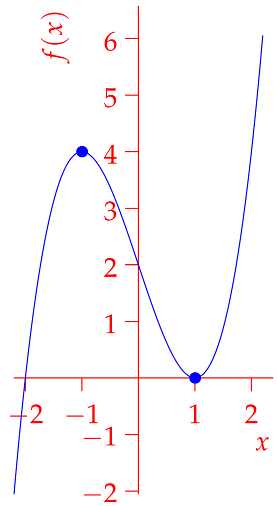
Example The function $f(x) = \begin{cases} \sqrt{x} & x \geq 0 \\ x & x < 0 \end{cases}$ is not differentiable at the origin, whence $x = 0$ is a critical value of f.
However, $(0, 0)$ is neither a local maximum nor minimum of the function.



Supposing that we ignore endpoints of graphs, we can summarize Fermat's Theorem as follows:

- Local max/min $\Longrightarrow$ Critical point
- Critical Point $\not\Longrightarrow$ Local max/min

Things are simplest for functions differentiable everywhere: we need only look for places where the derivative vanishes.



Example Find the local maxima and minima of

$$f(x) = x^3 - 3x + 2 \qquad \text{where } x \in \mathbb{R}$$

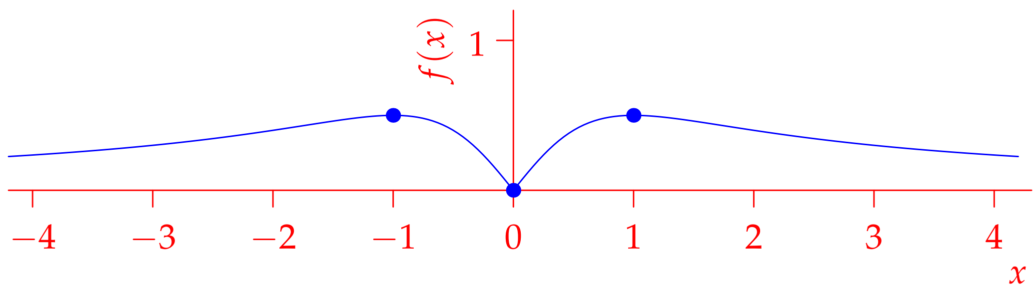
f is differentiable everywhere, with

$$f'(x) = 3x^2 - 3 = 0 \iff x = \pm 1$$

There are therefore two critical points: $(-1, 4)$ and $(1, 0)$.
Examining $f(x)$ when x is near ± 1 we see that these are local maximum and minimum points respectively.

For non-differentiable functions we need to be more careful.

Example The graph shows $f(x) = \dfrac{|x|}{1+x^2} = \begin{cases} \dfrac{x}{1+x^2} & \text{if } x \geq 0 \\ \dfrac{-x}{1+x^2} & \text{if } x < 0 \end{cases}$



f is differentiable when $x \neq 0$. Indeed for $x > 0$ we have

$$f'(x) = \frac{(1+x^2) - x \cdot 2x}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

Consider the derivative when $x < 0$ we obtain

$$f'(x) = \begin{cases} \dfrac{1-x^2}{(1+x^2)^2} & \text{if } x > 0 \\ \dfrac{x^2-1}{(1+x^2)^2} & \text{if } x < 0 \end{cases}$$

The critical values are $x = 0, \pm 1$, yielding the critical points $(0, 0)$, $(\pm 1, \frac{1}{2})$. These are a local minimum and local maxima respectively.

Closed Intervals

If the domain of f is a closed interval, we can often say more.

Theorem (Extreme Value). *If f is continuous on a closed bounded interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some values c and d between a and b.*

Combining this with Fermat's Theorem gives us a method for finding the absolute maximum and minimum values of a function f defined on an interval $[a, b]$:

1. Find the critical values $c_1, c_2, \ldots$ whenever $a < x < b$.
2. Compute $f(c_1), f(c_2), \ldots$
3. Compute $f(a)$ and $f(b)$.
4. Compare all the values of $f(x)$ in steps 2 and 3: the largest is the absolute maximum and the smallest the absolute minimum.

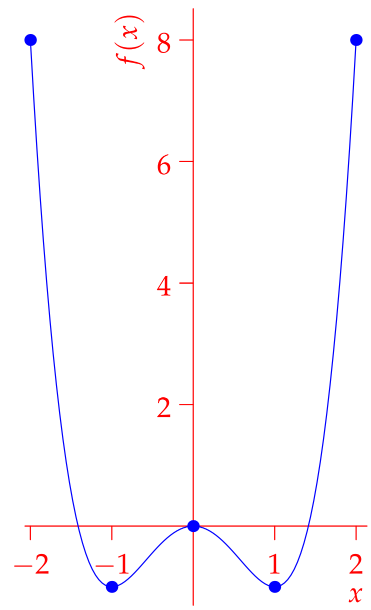
Example $f(x) = x^4 - 2x^2$ is continuous and differentiable on the closed interval $[-2, 2]$. We have

$$f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$$

1. There are three critical values: $x = 0, 1, -1$.
2. $f(0) = 0$, $f(1) = -1$ and $f(-1) = -1$.
3. At the endpoints we have $f(-2) = 8$ and $f(2) = 8$.

The maxima and minima of f are therefore:

Points	Type
$(-2, 8)$, $(2, 8)$	Absolute Maxima
$(-1, -1)$, $(1, -1)$	Absolute Minima
$(0, 0)$	Local Maximum



Homework

A Farmer sells x lb of strawberries at a cost of $c(x) = 10 - \frac{1}{20}x$ \$/lb. The Farmer wants to find what quantity of strawberries to sell in order to maximize his profit.

1. Explain why the *profit function*, the function the Farmer needs to maximize is

$$p(x) = xc(x) = 10x - \frac{1}{20}x^2 = \frac{1}{20}x(200 - x)$$

2. What quantity of strawberries should the farmer sell, and what is their profit.?