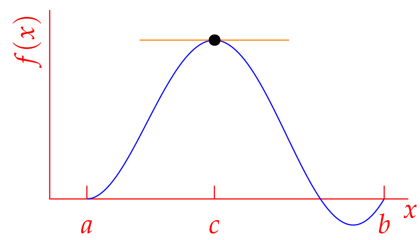
4.2 The Mean Value Theorem

The Mean Value Theorem is one of the most important results in calculus. We prove it as a consequence of a slightly simpler result.



Theorem (Rolle)**.** *Suppose that f is continuous on a closed interval $[a, b]$, differentiable on (a, b), and that $f(a) = f(b) = 0$. Then there exists some $c \in (a, b)$ for which $f'(c) = 0$.*

The essential idea is that if a differentiable function starts and finishes at the same value, and starts heading upwards, then at some point it must turn around and start heading down again.

Proof. Suppose that f is non-constant, for otherwise any c will do.

Without loss of generality, assume that f is positive somewhere. Since f is continuous, the Extreme Value Theorem says that f attains its maximum value $f(c)$ for some $c \in (a, b)$. Note that $c \neq a, b$ since f is positive somewhere.
Let $h \neq 0$ be small so that $c + h \in [a, b]$. Then $f(c + h) - f(c) \leq 0$ since $f(c)$ is the absolute maximum value of f. But then

$$\frac{f(c+h) - f(c)}{h} \quad \begin{cases} \leq 0 & \text{if } h > 0 \\ \geq 0 & \text{if } h < 0 \end{cases} \tag{$\dagger$}$$

Since f is differentiable, we note that

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

is equal to both the left and right limits of $(\dagger)$ at $h = 0$. However the right limit is necessarily ≤ 0 and the left limit ≥ 0. It follows that $f'(c) = 0$.

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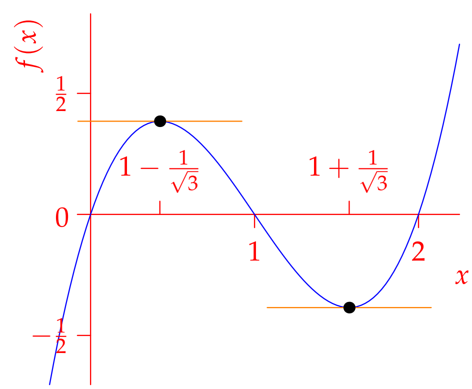
Corollary. *If f is a differentiable function, then between every pair of solutions to $f(x) = 0$ there is a solution to $f'(x) = 0$.*



Example $f(x) = x(x - 1)(x - 2)$ has three roots, $x = 0, \pm 1$.
The Corollary says that $f'(x) = 0$ has at least two roots.
In this case we may solve explicitly:

$$f(x) = x^3 - 3x^2 + 2x \implies f'(x) = 3x^2 - 6x + 2$$

$$f'(x) = 0 \iff x = \frac{6 \pm \sqrt{6^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} = 1 \pm \frac{1}{\sqrt{3}}$$

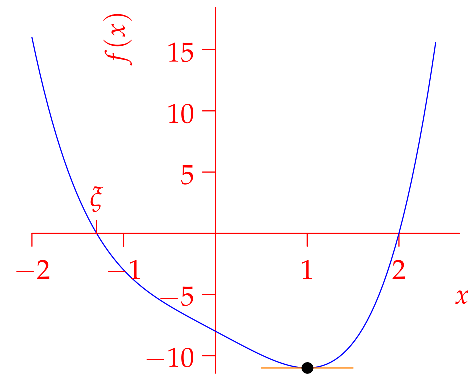
We can combine the Mean Value and Intermediate Value Theorems to tell us precisely how many roots a particular equation has.

Example If $f(x) = x^4 - 4x - 8$, how many roots has the equation $f(x) = 0$?

First differentiate:

$$f'(x) = 4x^3 - 4$$

$f'(x) = 0$ has only one root, $x = 1$. By the Corollary, $f(x) = 0$ has *at most* two roots.



Now we apply the Intermediate Value Theorem.

$$f(0) = -8 < 0 < 16 = f(-2) \implies \text{there is a root } \xi \text{ satisfying } -2 < \xi < 0$$

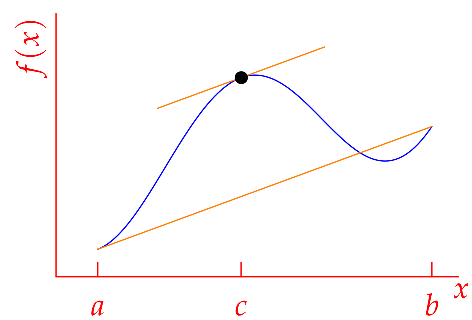
A second application of the Theorem, or simply spotting that $f(2) = 0$, shows that $f(x) = 0$ has *at least* two roots.
Combining the steps, we conclude that $x^4 - 4x - 8 = 0$ has *exactly* two roots.

The Mean Value Theorem proper, is simply Rolle's Theorem on a constant slope. It can be summarized as:

Average slope of f = instantaneous slope *somewhere.*

Theorem (Mean Value)**.** *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Proof. Let $g(x) = f(x) - f(a) - \dfrac{f(b) - f(a)}{b - a}(x - a)$.
g satisfies the hypotheses of Rolle's Theorem ($g(a) = g(b) = 0$) and so there exists $c \in (a, b)$ with $g'(c) = 0$. But then

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

which is the result. ∎

Example Find all the possible values $x = c$ satisfying the Mean Value Theorem on the interval $[-2, 6]$ for

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2 + x$$

We have

$$f'(x) = x^3 - 6x^2 + 8x + 1$$

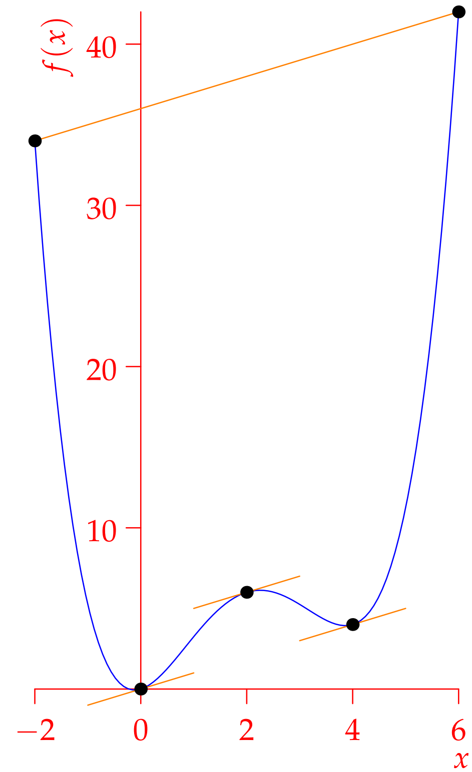
whence $f(-2) = 34$ and $f(6) = 42$. The average slope is therefore

$$\frac{f(6) - f(-2)}{6 - (-2)} = 1$$

We must therefore solve $f'(c) = 1$. But this is if and only if

$$0 = c(c^2 - 6c + 8) = c(c-2)(c-4)$$

whence $c = 0, 2,$ or 4.



How Many Functions have the *same* Derivative? One of the reasons that the Mean Value Theorem is so important to calculus is the fact that it answers the above question.

Theorem. *If $f'(x) = g'(x)$ on some interval I, then $f(x) - g(x)$ is constant on I.*

Proof. Let $h(x) = f(x) - g(x)$ and let $a < b$ in I. h satisfies the Mean Value Theorem on $[a, b]$, hence there exists $c \in (a, b)$ with

$$\frac{h(b) - h(a)}{b - a} = h'(c) = f'(c) - g'(c) = 0$$

But then $h(b) = h(a)$ for all $a, b \in I$. Hence h is constant on I. ∎

Example Find all functions $f(x)$ such that $f'(x) = \cos x + 2x$ for all $x \in \mathbb{R}$.

We know that $\frac{d}{dx}(\sin x + x^2) = \cos x + 2x$, thus $f(x)$ must have the form $f(x) = \sin x + x^2 + C$ for some constant C.

Corollary. *$f' = 0$ on an interval $\Longrightarrow f$ constant.*

Homework

In each case below, sketch and come up with a formula of an example function which *fails* to satisfy the conclusion of the Mean Value Theorem. That is for which there are no values of $c \in (a, b)$ satisfying

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

1. f is not continuous on $[a, b]$ but is differentiable on (a, b).
2. f is continuous on $[a, b]$ but isn't differentiable on (a, b).