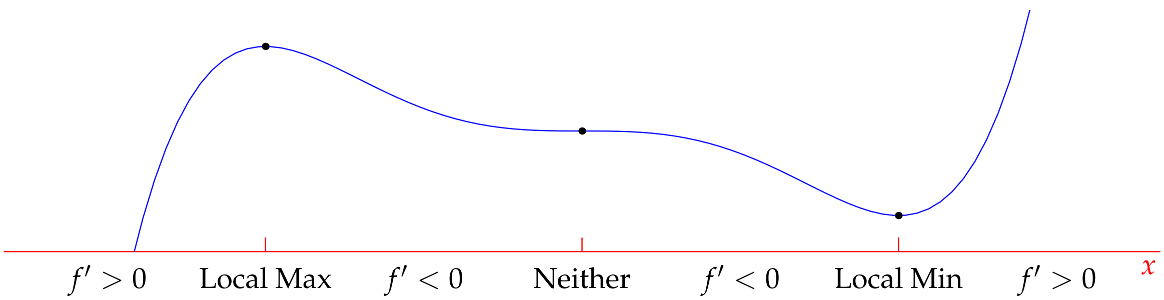
4.3 How Derivatives affect the Shape of a Graph

What do f' and f'' say about f?

Increasing/Decreasing If $f' > 0$ on an interval, then f is *increasing* on that interval.
Similarly $f' < 0 \implies f$ decreasing.

Theorem (First Derivative Test)**.** *Suppose that c is a critical value of a continuous function f.*

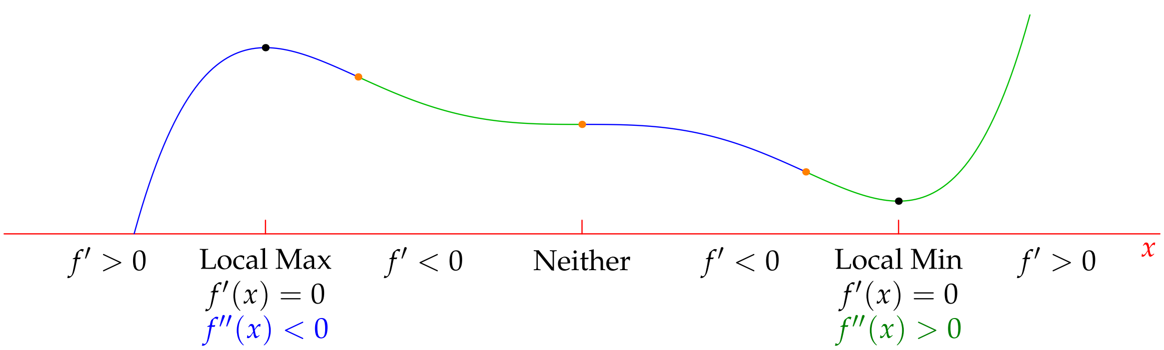
- *If f' changes from +ve to -ve at c then f has a local maximum at c.*
- *If f' changes from -ve to +ve at c then f has a local minimum at c.*
- *If f' does not change sign then f has neither a local maximum nor minimum at c.*



Concavity If $f'' > 0$ on an interval, then f is said to be *concave upwards* on that interval.
Similarly $f'' < 0 \implies f$ is concave downwards.

Theorem (Second Derivative Test)**.** *Suppose that f is continuous near $x = c$.*

- *If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.*
- *If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.*



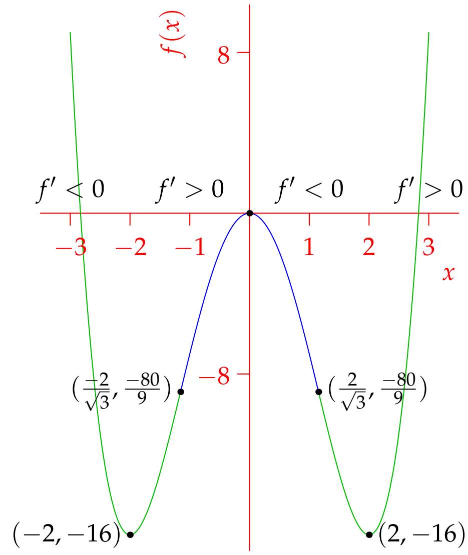
Definition. *A curve $y = f(x)$ has an* inflection point *at $x = c$ if the curve is continuous at c and changes from concave upward to concave downward at c (or vice versa).*

If f is twice continuously differentiable at an inflection point $x = c$, then we necessarily have $f''(c) = 0$.

If the second derivative test produces $f'(c) = 0$ and $f''(c) = 0$ then f has either a local minimum, a maximum or an inflection point at c: the test is inconclusive. We must either investigate the concavity on either side of $x = c$ or use the first derivative test. The graph above has three inflection points.

Examples

1. Consider $f(x) = x^4 - 8x^2$. Differentiate and set equal to zero to find the critical values:



$$f'(x) = 4x^3 - 16x = 4x(x-2)(x+2)$$
$$= 0 \iff x = 0, \pm 2$$

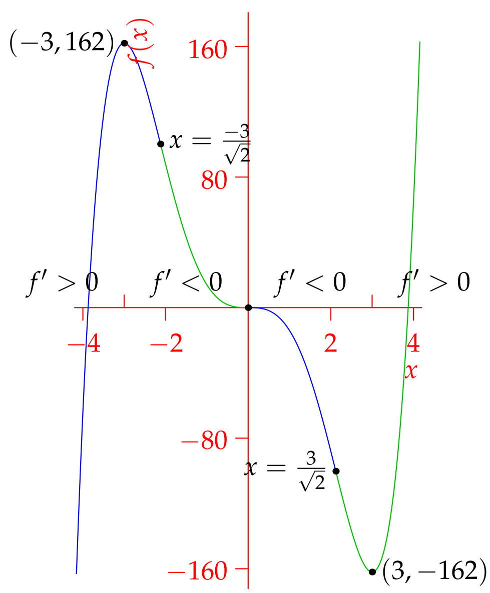
The critical points are $(0, 0)$, $(2, -16)$, $(-2, -16)$. We still need to consider the *sign* of the derivative between the critical values in order to tell when the function is increasing/decreasing.
Now search for inflection points:

$$f''(x) = 12x^2 - 16 = 4(3x^2 - 4)$$
$$= 0 \iff x = \pm\frac{2}{\sqrt{3}}$$

We can summarize in a table:

Interval	f'	f	Interval	f''	Concavity
$(-\infty, -2)$	$-$	decreasing	$\left(-\infty, \frac{-2}{\sqrt{3}}\right)$	$+$	upward
$(-2, 0)$	$+$	increasing	$\left(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$	$-$	downward
$(0, 2)$	$-$	decreasing	$\left(\frac{2}{\sqrt{3}}, \infty\right)$	$+$	upward
$(2, \infty)$	$+$	increasing			

2. Repeat with $f(x) = x^5 - 15x^3$.



$$f'(x) = 5x^4 - 45x^2 = 5x^2(x-3)(x+3)$$
$$= 0 \iff x = 0, \pm 3$$

Critical points: $(0, 0)$, $(3, -162)$, $(-3, 162)$.

$$f''(x) = 20x^3 - 90x = 10x(2x^2 - 9)$$
$$= 0 \iff x = 0, \pm\frac{3}{\sqrt{2}}$$

In summary:

Interval	f'	f	Interval	f''	Concavity
$(-\infty, -3)$	$+$	inc	$\left(-\infty, \frac{-3}{\sqrt{2}}\right)$	$-$	down
$(-3, 0)$	$-$	dec	$\left(\frac{-3}{\sqrt{2}}, 0\right)$	$+$	up
$(0, 3)$	$-$	dec	$\left(0, \frac{3}{\sqrt{2}}\right)$	$-$	down
$(3, \infty)$	$+$	inc	$\left(\frac{3}{\sqrt{2}}, \infty\right)$	$+$	up