4.4 Indeterminate Forms and l'Hôpital's Rule

The limit laws seem straightforward: it appears that much of the time we can compute $\lim_{x\to a} F(x)$ by evaluation of all the piece of $f(x)$ at $x = a$. Occasionally there are problems. For example:

1. May have to cancel factors: $\lim_{x\to 1} \frac{x^2-1}{x-1} \overset{????}{=} \frac{0}{0}$ is meaningless. Instead

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1} x+1 = 2$$

2. With $\lim_{x\to 0} \frac{\sin x}{x} = 1$ we had to argue geometrically.

These limits are examples of *indeterminate forms:* expressions where evaluating the limit by substitution results in a meaningless mathematical expression such as $\frac{0}{0}$. There are several other such expressions.

Definition. *An* indeterminate form *is a limit* $\lim_{x\to a} F(x)$*, where evaluating* $F(a)$ *directly gives one of the meaningless expressions*

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 0^\infty \qquad \infty^0 \qquad 1^\infty$$

For example, $\lim_{x\to 0} \frac{\sin x - x}{\tan x + x}$ produces the indeterminate form $\frac{0}{0}$. How should we deal with this limit?

Theorem (l'Hôpital's Rule[1]). *Let* f, g *be differentiable and* $g'(x) \neq 0$ *on an open interval containing* a*, but possibly not at* a*. Suppose that* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *is an indeterminate form of type* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$*. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the latter limit exists, or is $\pm\infty$*. The rule also applies to one-sided limits and limits at infinity.*

Examples

1. $\lim_{x\to 0} \frac{\sin x - x}{\tan x + x} = \lim_{x\to 0} \frac{\cos x - 1}{\sec^2 x + 1} = \frac{1-1}{1+1} = 0$ (type $\frac{0}{0}$)

2. $\lim_{x\to \pi/2} \frac{x-\pi/2}{\cos x} = \lim_{x\to \pi/2} \frac{1}{-\sin x} = -1$ (type $\frac{0}{0}$)

3. $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}\lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}$ (type $\frac{0}{0}$)

4. Here we apply l'Hôpital's Rule before having to simplify

$$\lim_{x\to\infty} \frac{\ln x}{\ln(1+x^2)} = \lim_{x\to\infty} \frac{1/x}{2x/(1+x^2)} = \lim_{x\to\infty} \frac{1+x^2}{2x^2} = \frac{1}{2} \qquad \text{(type } \tfrac{\infty}{\infty}\text{)}$$

[1]Named for the Marquis de l'Hôpital, a French nobleman of the 17th to early 18th centuries. A circumflex in modern French denotes where a silent s used to follow the o, hence why you sometimes see this as 'hospital's rule.' The ô is pronounced like the o in 'go'—not hospital!

5. This example requires *two* applications of l'Hôpital's Rule

$$\lim_{x\to\infty}\frac{e^{x^2}}{x^3}=\lim_{x\to\infty}\frac{2xe^{x^2}}{3x^2}=\lim_{x\to\infty}\frac{2e^{x^2}}{3x}=\lim_{x\to\infty}\frac{4xe^{x^2}}{3}=\infty \qquad \left(\text{type } \tfrac{\infty}{\infty}\right)$$

Warnings! The Rule is almost *too* easy to use; it is therefore very easy to *misuse.* Here are some common mistakes/issues.

- Don't use the quotient rule! Differentiate f and g *separately.*
- *Simplify* before applying! For instance

  $$\lim_{x\to\infty}\frac{e^x}{e^{2x}+e^{3x}}=\lim_{x\to\infty}\frac{e^x}{2e^{2x}+3e^{3x}}=\lim_{x\to\infty}\frac{e^x}{4e^{2x}+9e^{3x}}=\cdots$$

  is an indeterminate form of type $\frac{\infty}{\infty}$. Applying l'Hôpital's rule unthinkingly results in a never-ending chain of limits as above. Instead just factorize: the rule is not required!

  $$\lim_{x\to\infty}\frac{e^x}{e^{2x}+e^{3x}}=\lim_{x\to\infty}\frac{1}{e^x+e^{2x}}=0$$

- *Only* applies to indeterminate forms of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. For example,

  $$\lim_{x\to 1}\frac{x}{1-x}\neq\lim_{x\to 1}\frac{1}{-1}=-1$$

  The left hand side is *not* and indeterminate form, so l'Hôpital's rule does not apply. Indeed the original limit does not exist.

- It is possible for $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ to exist, even when $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ does not. In such a case, the Rule does not apply. For example,

  $$\lim_{x\to\infty}\frac{x+\cos x}{x}$$

  is an indeterminate form of type $\frac{\infty}{\infty}$. If we try to apply the Rule, we obtain

  $$\lim_{x\to\infty}\frac{x+\cos x}{x}\stackrel{?}{=}\lim_{x\to\infty}\frac{1-\sin x}{1}=\text{DNE}$$

  This is an incorrect use of the Rule. Re-read the statement of the Rule: it only applies *if* the limit $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ exists. In this case the second limit does not exist, whence we cannot use the rule. This example can be found much more easily by elementary methods:

  $$\lim_{x\to\infty}\frac{x+\cos x}{x}=\lim_{x\to\infty}1+\frac{\cos x}{x}=1$$

- In a logical sense, the following application of l'Hôpital's rule is incorrect:

  $$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1$$

  The objection is one of circular logic: the only reason we wanted to compute this limit was to *prove* that the derivative of sine is cosine. If we use this to compute the limit, our logic is circular! This doesn't mean that the rule isn't a useful way of reminding yourself of this result if you're stuck!

Indeterminate Products, Differences and Powers

The remaining indeterminate forms mentioned earlier can all be attacked using l'Hôpital's rule, after some algebraic manipulation.

Definition. *An* indeterminate product *is a limit* $\lim_{x\to a} f(x)g(x)$ *where* $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = \infty$.

These can be tackled as follows:

$$\lim_{x\to a} \frac{1}{g(x)} = 0 \implies \lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{1/g(x)} \quad \text{which is now an indeterminate form of type } \tfrac{0}{0}$$

Now apply l'Hôpital's Rule as before. You could also consider $\lim_{x\to a} \frac{g(x)}{1/f(x)}$ of type $\frac{\infty}{\infty}$.

Example

$$\begin{aligned}\lim_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)\tan x &= \lim_{x\to\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{1/\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{-1}{-\sec^2 x/\tan^2 x} \\ &= \lim_{x\to\frac{\pi}{2}} \cos^2 x \tan^2 x = \lim_{x\to\frac{\pi}{2}} \sin^2 x = 1\end{aligned}$$

Definition. *An* indeterminate difference *is a limit* $\lim_{x\to a}(f(x) - g(x))$ *where* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$

The approach is to try combine $f(x) - g(x)$ into a single fraction over a common denominator: this will typically yield an indeterminate form of a simpler type.

Example

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x} - \csc x\right) &= \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0} \frac{\sin x - x}{x \sin x} && \text{(combine to a single fraction of type } \tfrac{0}{0}\text{)} \\ &= \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x\cos x} && \text{(apply l'Hôpital's rule, still type } \tfrac{0}{0}\text{)} \\ &= \lim_{x\to 0} \frac{-\sin x}{2\cos x - x\sin x} && \text{(apply l'Hôpital's rule again)} \\ &= \frac{0}{2-0} = 0\end{aligned}$$

Definition. *An* indeterminate power *is a limit* $\lim_{x\to a} f(x)^{g(x)}$ *where* $f(a)^{g(a)}$ *would yield* 0^0, ∞^0, *or* 1^∞

These are dealt with similarly to logarithmic differentiation. Since the natural exponential is a continuous functions we can take them through the limit operator:

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} \exp(g(x)\ln f(x)) = \exp\left(\lim_{x\to a} g(x)\ln f(x)\right)$$

This reduces the problem to that of finding the limit of an indeterminate product:

Indeterminate Form $f(a)^{g(a)}$	0^0	∞^0	1^∞
Indeterminate Product $g(a)\ln f(a)$	$0\cdot(-\infty)$	$0\cdot\infty$	$\infty\cdot 0$

Examples

1. $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} \exp(x\ln x) = \exp\left(\lim_{x\to 0^+} x\ln x\right)$

   Use l'Hôpital to find

   $$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

   from which $\lim_{x\to 0^+} x^x = e^0 = 1$.

2. $\lim_{x\to 0^+} (\sin x)^{\sin x} = \lim_{x\to 0^+} e^{\sin x \ln \sin x} = \exp\left(\lim_{x\to 0^+} \sin x \ln \sin x\right)$

   Use l'Hôpital to find

   $$\begin{aligned}\lim_{x\to 0^+} \sin x \ln \sin x &= \lim_{x\to 0^+} \frac{\ln \sin x}{\csc x} = \lim_{x\to 0^+} \frac{\cos x/\sin x}{-\cos x/\sin^2 x} \\ &= \lim_{x\to 0^+} (-\sin x) = 0\end{aligned}$$

   from which $\lim_{x\to 0^+} (\sin x)^{\sin x} = e^0 = 1$.

3. $\lim_{x\to 0^+} (1+\sin x)^{\csc x} = \lim_{x\to 0^+} \exp(\csc x \ln(1+\sin x))$

   Now use l'Hôpital

   $$\begin{aligned}\lim_{x\to 0^+} \csc x \ln(1+\sin x) &= \lim_{x\to 0^+} \frac{\ln(1+\sin x)}{\sin x} = \lim_{x\to 0^+} \frac{\cos x/(1+\sin x)}{\cos x} \\ &= \lim_{x\to 0^+} \frac{1}{1+\sin x} = 1\end{aligned}$$

   from which $\lim_{x\to 0^+} (1+\sin x)^{\csc x} = e^1 = e$.

4. $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} \exp\left(\tfrac{1}{x}\ln x\right) = \exp\left(\lim_{x\to\infty} \tfrac{1}{x}\ln x\right)$

   Use l'Hôpital to find

   $$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

   from which $\lim_{x\to\infty} x^{1/x} = e^0 = 1$.