4.5 Summary of Curve Sketching

When graphing a function f you want to make clear all of the following, if they make sense for the function.

Domain Interval? Excluded points? If $\text{dom}(f) = \mathbb{R}$, make sure it's clear what happens for very large values of x.

Intercepts Find the x- and y-intercepts, if appropriate.

Symmetry f could have various types of symmetry, or none:

- Periodicity: is there some constant c such that $f(x+c) = f(x)$ for all x?
- Is f odd? ($f(-x) = -f(x)$, graph $180°$ rotationally symmetric about the origin)
- Or is f even ($f(-x) = f(x)$, graph has reflection symmetry across y-axis)

Asymptotes Vertical and horizontal, if appropriate.

Critical Points When is $f'(c) = 0$ or undefined?

Intervals of Inc/Dec When is $f'(x)$ positive/negative?

Local max/min Apply the 1st or 2nd derivative test.

Concavity and Points of Inflection When is $f''(x)$ positive/negative, and when does it change?

Example 1. Graph $y = f(x) = \frac{x^2+x}{x^2+x-2} = \frac{x(x+1)}{(x-1)(x+2)}$

Domain $\mathbb{R} \setminus \{-2, 1\}$: moreover $\lim\limits_{x\to\pm\infty} y = 1$

Intercepts $y = 0 \implies x = -1, 0$ and $x = 0 \implies y = 0$

Symmetry None

Asymptotes Horizontal $y = 1$, Vertical $x = 1, -2$

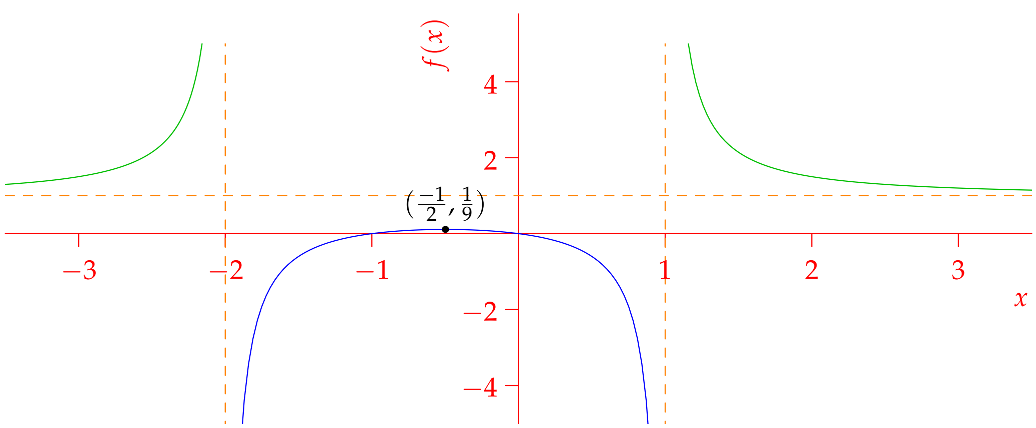
Critical Points $f'(x) = \frac{-2(2x+1)}{(x^2+x-2)^2}$. Zero at $x = \frac{-1}{2}$.

Increase/Decrease f increases when $x < -\frac{1}{2}$, $\quad f$ decreases when $x > -\frac{1}{2}$

Local max/min Critical point $(\frac{-1}{2}, \frac{1}{9})$ is a local maximum by the first derivative test

Concavity $f''(x) = \frac{12(x^2+x+1)}{(x^2+x-2)^3} = \frac{12(x^2+x+1)}{(x-1)^3(x+2)^3} = \frac{12((x+\frac{1}{2})^2+\frac{3}{4})}{(x-1)^3(x+2)^3}$

Concave up for $x < -2$ or $x > 1$, concave down otherwise



Example 2. Graph $y = \sin x + x$

Domain $\mathbb{R}$: moreover $\lim_{x\to\pm\infty} y = \pm\infty$

Intercepts $y = 0 \iff x = 0$: crosses axes only at the origin

Symmetry Function odd

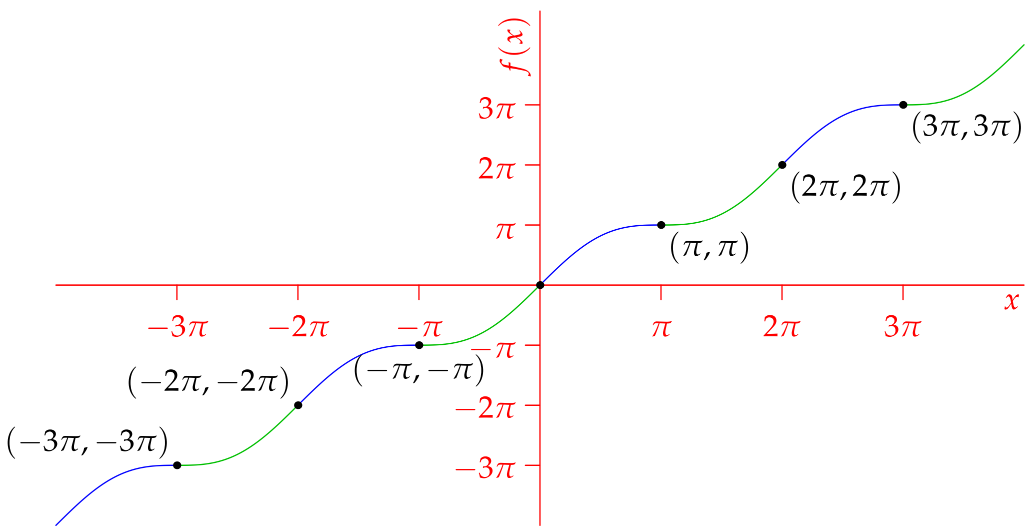
Asymptotes None

Critical Points $f'(x) = \cos x + 1$. Zero at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$

Inc/Decrease f increases when $\cos x \neq -1$ which is everywhere *except* at the critical points.
f never decreases.

Local max/min 1st derivative test $\implies$ none of the critical points are local maxima or minima.

Concavity $f''(x) = -\sin x$.
Concave up if $(2n+1)\pi < x < 2n\pi$, where n is any integer
Concave down if $2n\pi < x < (2n+1)\pi$
Inflection points $(n\pi, n\pi)$ for each integer n



Example 3. Graph $y = x^4 - 3x^2 + 2x = x(x^3 - 3x + 2) = x(x-1)^2(x+2)$

Domain $\mathbb{R}$: moreover $\lim_{x \to \pm\infty} y = \infty$

Intercepts $y = 0 \implies x = 0, 1, -2$ and $x = 0 \implies y = 0$

Symmetry None

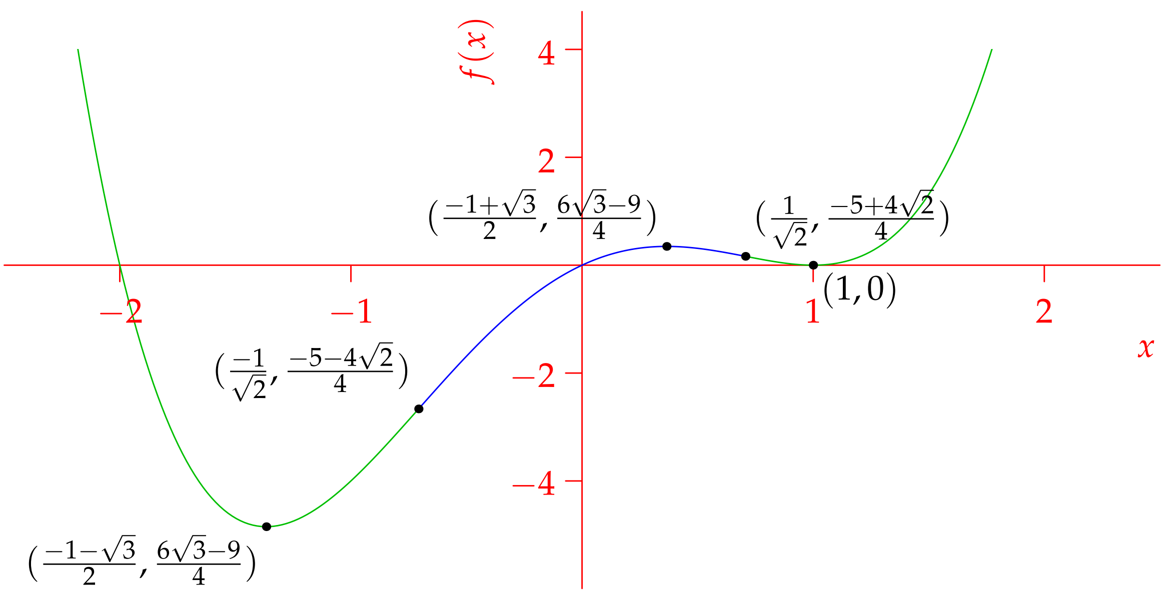
Asymptotes None

Critical Points $f'(x) = 4x^3 - 6x + 2 = 2(x-1)(2x^2 + 2x - 1)$. Quadratic formula gives zeros at $x = 0, \frac{-1\pm\sqrt{3}}{2}$.

Inc/Decrease Since $f'(x) = 0$ is *cubic* with three distinct zeros, $f'(x)$ must change sign at each of its zeros. Placing these in order $\frac{-1-\sqrt{3}}{2} < \frac{-1+\sqrt{3}}{2} < 1$ and noting that f' is positive for x large, we see that:
f increases when $\frac{-1-\sqrt{3}}{2} < x < \frac{-1+\sqrt{3}}{2}$ or $1 < x$
f decreases when $x < \frac{-1-\sqrt{3}}{2}$ or $\frac{-1+\sqrt{3}}{2} < x < 1$

Local max/min Critical points: $(1, 0)$ is a local minimum by the 1st derivative test
$(\frac{-1+\sqrt{3}}{2}, \frac{-6\sqrt{3}-9}{4})$ is a local maximum[1]
$(\frac{-1-\sqrt{3}}{2}, \frac{6\sqrt{3}-9}{4})$ is a local minimum

Concavity $f''(x) = 12x^2 - 6 = 12(x - \frac{1}{\sqrt{2}})(x + \frac{1}{\sqrt{2}})$
Concave up if $x < \frac{-1}{\sqrt{2}}$, or $x > \frac{-1}{\sqrt{2}}$
Concave down if $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$
Inflection points $(\frac{-1}{\sqrt{2}}, \frac{-5-4\sqrt{2}}{4})$, $(\frac{1}{\sqrt{2}}, \frac{-5+4\sqrt{2}}{4})$

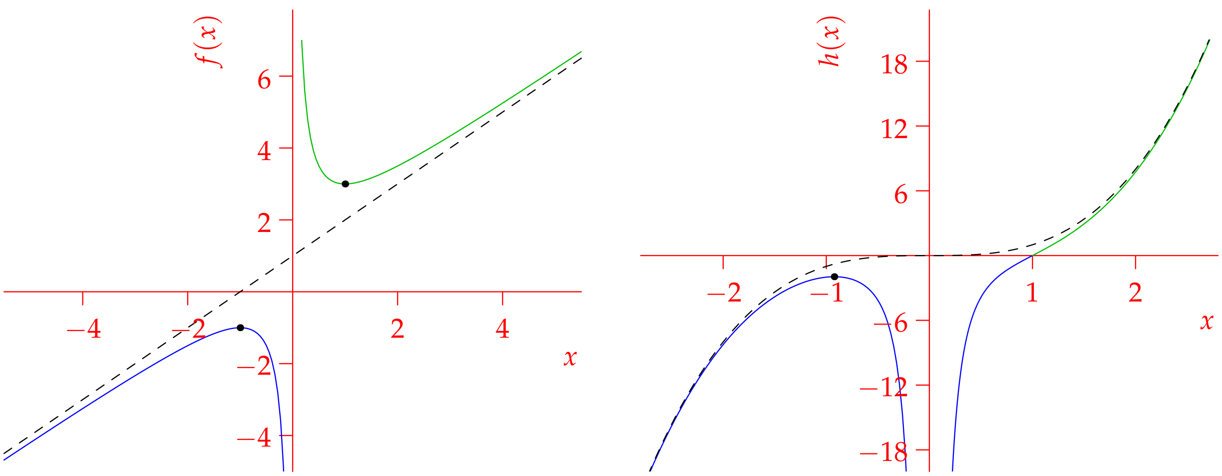


[1] It may not be worth computing the y-co-ordinate in a test if this will take a long time.

Slant Asymptotes One last idea property to look out for are *slant asymptotes.* A curve $y = f(x)$ may get arbitrarily close to another curve $y = g(x)$ as $x \to \pm\infty$: in such a case we say that f is *asymptotic to* g. When the graph of g is a straight line, we call this a slant asymptote of f.

Examples

1. $f(x) = \frac{x^2+x+1}{x} = x + 1 + \frac{1}{x}$ is asymptotic to $g(x) = x + 1$ (since $\lim_{x \to \pm\infty} \frac{1}{x} = 0$). Therefore $y = x + 1$ is a slant asymptote of f.
2. $h(x) = x^3 - \frac{1}{x^2}$ is asymptotic to $y = x^3$.



Homework

Try sketch each of the following functions:

1. $y = x^4 + 8x^3 - 270x^2 + 1$
2. $y = \frac{4x+4}{x^2+3}$
3. $y = \frac{x^2}{x^2-4}$
4. $y = \frac{\ln x}{x^2}$