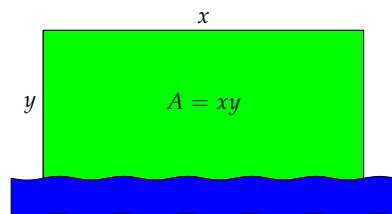## 4.7 Optimization Problems

We use calculus to find the the optimal solution to a problem: usually this involves two steps.

1. Convert a word problem into the form 'Find the maximum/minimum value of a function.' This is often the hard part as the word problem may not have any equations or variable, so you might have to *invent* your own. Draw a picture if it is helpful.

2. Solve the max/min problem and interpret your answer. Some of the solutions may not be physically relevant.

Example 1. A three-sided fence is to be built along the edge of a river to make a rectangular field. Find the maximum area of the field if there is $50\,\mathrm{m}$ of fencing available.

- Introduce variables: $x = $ length parallel to river, $y = $ width of field.

- Fence length $= 50 = x + 2y \implies y = 25 - \frac{1}{2}x$.
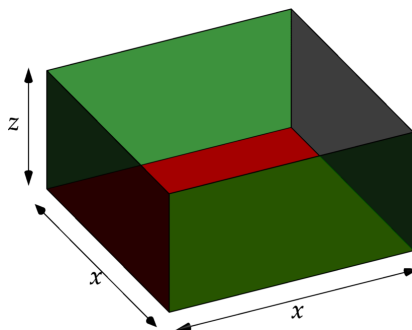
- Area $A(x) = xy = x(25 - \frac{1}{2}x)$.

The problem can now be rephrased: Maximise the function $A(x) = x(25 - \frac{1}{2}x)$ where[1] $0 \le x \le 50$. Differentiate to find the only critical value

$$A'(x) = 25 - x = 0 \iff x = 25$$

Either observe that $y = A(x)$ is a downward opening parabola or use a derivative test to see that we have a maximum at $x = 25\,\mathrm{m}$. The largest possible field therefore has dimensions $x = 25\,\mathrm{m}$, $y = 12.5\,\mathrm{m}$, and area $A(25) = 312.5\,\mathrm{m}^2$.

Example 2. A cuboidal box of volume $4\,\mathrm{ft}^3$ is to be made with a square base and no lid: find the dimensions of the box with minimal surface area.

- Label the edges x, x, and z.

- Volume: $V = x^2 z = 4 \implies z = 4x^{-2}$.

- Surface Area: $A(x) = x^2 + 4xz = x^2 + 16x^{-1}$.

To minimize A we differentiate:

$$A'(x) = 2x - 16x^{-2} = \frac{2(x^3 - 8)}{x^2} \implies \text{critical values } x = 0, 2$$

Only $x > 0$ is physically relevant[2] so so we ignore $x = 0$. The second derivative test now says,

$$A''(x) = 2 + 32x^{-3} \implies A''(2) > 0 \quad \text{so } A(2) \text{ is a local minimum.}$$

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[1] The minimum value of x is clearly zero, giving a field with no width and therefore no area! These restrictions aren't strictly necessary, but it is important to note, in general, which values of your variables give physically reasonable solutions. Here, for instance, if $x > 50$, then the field has negative area: clearly an absurdity!

[2] This time $x = 0$ is unphysical, requiring an infinitely tall box if the volume was still to be $4\,\mathrm{ft}^3$.

Moreover, $\lim\limits_{x\to 0^+,\infty} A(x) = \infty$, so $A(2) = 12\,\text{ft}^2$ is the absolute minimum surface area.

The dimensions of the required box are therefore $x = 2\,\text{ft}$, and $z = 1\,\text{ft}$.

Example 3. Suppose that the box in Example 2 is made of material costing $\$2/\text{ft}^2$ for the base and $\$1/\text{ft}^2$ for the sides. What dimensions produce the cheapest box?

We still have $V = x^2 z = 4 \implies z = 4x^{-2}$, but now we must minimize the cost $C(x)$ of the box as a function of x. We first compute $C(x)$:

$$C(x) = (\text{number of sides}) \cdot (\text{area per side}) \cdot (\text{cost of side per unit area})$$
$$+ (\text{area of base}) \cdot (\text{cost of base per unit area})$$
$$= 4 \cdot xz \cdot 1 + x^2 \cdot 2 = 4xz + 2x^2$$
$$= 16x^{-1} + 2x^2 \qquad\qquad (\text{since } z = 4x^{-2})$$

Now find the critical points:

$$C'(x) = -16x^{-2} + 4x = 4x^{-2}(x^3 - 4)$$

which has critical values $x = 0, \sqrt[3]{4}$. Again $x = 0$ is unphysical, and

$$C''(x) = 32x^{-3} + 4 \implies C''(\sqrt[3]{4}) > 0$$

so $C(\sqrt[3]{4})$ is a local minimum. Since $\lim\limits_{x\to 0^+,\infty} C(x) = \infty$, it follows that $C(\sqrt[3]{4}) = \$6 \cdot 4^{2/3} \approx \$15.12$ is the absolute minimum cost.

The required box has dimensions $x = z = \sqrt[3]{4}\,\text{ft}$.

If we compare with the solution to Example 2., we obtain:

Box	Surface Area (ft^2)	Cost (\$)
Ex 2. (minimal area)	12	16
Ex 3. (minimal cost)	$5 \cdot 4^{2/3} \approx 12.599$	$6 \cdot 4^{2/3} \approx 15.12$

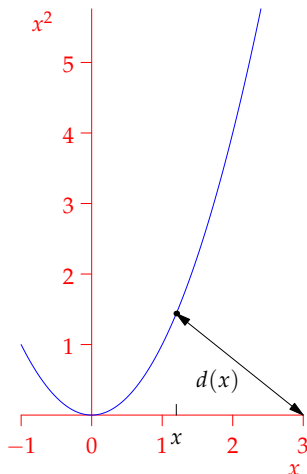**Example 4.** Find the point on the curve $y = x^2$ which is closest to $(3, 0)$.

Minimizing the distance is equivalent to minimizing the distance-squared (when one is minimal, so is the other!), so we choose to minimize the function

$$D(x) := d(x)^2 = (x - 3)^2 + (y - 0)^2$$
$$= (x - 3)^2 + (x^2 - 0)^2$$
$$= (x - 3)^2 + x^4$$

Now differentiate:

$$D'(x) = 2(x - 3) + 4x^3 = 2(2x^3 + x - 3) = 2(x - 1)(2x^2 + 2x + 3)$$

Computing the discriminant $b^2 - 4ac = 4 - 4 \cdot 2 \cdot 3 = -20 < 0$ (quadratic formula), or completing

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the square for the quadratic factor, we see that the only real solution to $D'(x) = 0$ is $x = 1$. The distance-squared function has only one critical value, yielding the point $(1, 1)$ on the curve. Moreover $D''(x) = 12x^2 + 2 > 0$ so that $x = 1$ minimizes the distance. It wasn't asked for, but the minimal distance itself is

$$d(1) = \sqrt{D(1)^2} = \sqrt{(1-3)^2 + 1^4} = \sqrt{5}$$

Example 5. Find the points on the curve $y = x^2$ closest to $(0, 3/2)$

Again we minimize the distance-squared function:

$$D(x) := d(x)^2 = (x - 0)^2 + (y - 3/2)^2 = x^2 + \left(x^2 - \frac{3}{2}\right)^2$$
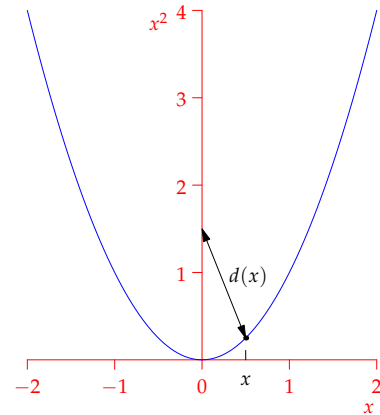
This time we have

$$D'(x) = 2x + 4x\left(x^2 - \frac{3}{2}\right) = 4x(x^2 - 1)$$

$$= 0 \iff x = 0, \pm 1$$

There are three critical points, corresponding to distances

$$d(\pm 1) = \frac{\sqrt{5}}{4} \quad \text{and} \quad d(0) = \frac{3}{2}$$

Moreover $D''(x) = 12x^2 - 4$ so that $D''(\pm 1) > 0$ and $D''(0) < 0$

It follows that $x = \pm 1$ give local minimum distances and $x = 0$ a local maximum distanct: the closest points on the curve are therefore $(\pm 1, 1)$.

Example 6. An object is launched at v m/s at an angle θ radians above the horizontal: its path follows the parbola with equation

$$y = \frac{x}{2\cos^2\theta}\left(\sin 2\theta - \frac{g}{v^2}x\right)$$

Which angle θ gives the maximum distance R?

First find the distance: the object hits the ground when

$$y = 0 \iff x = 0, \frac{v^2}{g}\sin 2\theta$$

We therefore wish to maximize the function $R(\theta) = \frac{v^2}{g}\sin 2\theta$. It's critical points occur when

$$0 = \frac{dR}{d\theta} = \frac{2v^2}{g}\cos 2\theta \iff 2\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots \iff \theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \ldots$$

Only angles $0 \leq \theta << \frac{\pi}{2}$ make physical sense, both resulting in zero distance, hence $\theta = \frac{\pi}{4}(= 45°)$ gives the maximum distance $R = \frac{v^2}{g}$.

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