4.9 Anti-derivatives

Anti-differentiation is exactly what it sounds like: the opposite of differntiation. That is, given a function f, can we find a function F whose derivative is f.

Definition. *An* anti-derivative *of a function f is a function F such that $F'(x) = f(x)$ for all x.*

Examples

1. $F(x) = \sin x$ is an anti-derivative of $f(x) = \cos x$.
2. $F(x) = (x^2+3)^{3/2}$ is an anti-derivative of $f(x) = 3x\sqrt{x^2+3}$.
3. $F(x) = \dfrac{x^2-3}{\cos x}$ is an anti-derivative of $f(x) = \dfrac{2x + (x^2-3)\tan x}{\cos x}$.

We can easily check the veracity of these statements by differentiating $F(x)$. But what if you are asked to *find* an anti-derivative, not just *check* something you've been given? In general this is a *hard* problem.[1] The *only* method that really exists for explicitly computing anti-derivatives is *guess and differentiate!* Indeed every famous rule that you'll study in Integration (substitution, integration by parts, etc.) is merely the result of guessing a general anti-derivative and checking that your guess is correct!

Anti-derivatives of Common Functions Guessing is, of course, easier if you have familiarity with differentiation. With each of the following functions $f(x)$ you should be able to guess the chosen anti-derivative $F(x)$ just from what you know about derivatives.

Function $f(x)$	Anti-derivative $F(x)$
$kf(x), \quad k$ constant	$kF(x)$
$f(x)+g(x)$	$F(x)+G(x)$
$x^n, \quad n \neq -1$	$\dfrac{1}{n+1}x^{n+1}$
x^{-1}	$\ln\|x\|$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
e^{kx}	$\frac{1}{k}e^{kx}$

Examples

1. $f(x) = 3 + \frac{2}{x}$ has an anti-derivative $F(x) = 3x + 2\ln|x|$.
2. $f(x) = 3\sec^2 x - 2e^{3x}$ has an anti-derivative $F(x) = 3\tan x - \frac{2}{3}e^{3x}$.

[1] Differentiation is often described as *easy* in the sense that nice functions can usually be differentiated via our familiar rules. Anti-differentiation is *hard:* very few functions have anti-derivatives that can easily be computed. For example, it is easy to find the derivative of $f(x) = e^{-x^2}$, namely $f'(x) = -2xe^{-x^2}$, but can you find an *anti-derivative* of $f(x)$? If, by this, you mean an *explicit* formula involving simple functions (algebraic, trigonometric, exponential), then the answer is no. The best we can do is to *prove* that an anti-derivative exists and then estimate it to whatever degree of accuracy we need.

3. Find an anti-derivative of $g(x) = \cos x + \dfrac{x^3 + \sqrt{x}}{x^2}$.

   The trick is to manipulate $g(x)$ so that its terms are in the table. We know that $\sin x$ is an anti-derivative of $\cos x$, so the challenge is to deal with the second term. Thankfully we can rewrite it as a sum of powers of x:

   $$\frac{x^3 + \sqrt{x}}{x^2} = x + x^{-3/2}$$

   which can now be anti-differentiated. It follows that a suitable anti-derivative of $g(x)$ is

   $$G(x) = \sin x + \frac{1}{2}x^2 - 2x^{-1/2}$$

How many anti-derivatives does a function have?

The definition of anti-derivative does nothing more than providing an alternative way of refering to a pair of functions f and F satisfying the equation $f(x) = F'(x)$. That is, the following statements mean *exactly the same thing.*

- $f(x)$ is *the* derivative of $F(x)$.
- $F(x)$ is *an* anti-derivative of $f(x)$.

The crucial observation is the difference between the articles *the* and *an:* a given function has at most one derivative, but (potentially) many anti-derivatives. Indeed, $F(x) = \sin x$ and $G(x) = 17 + \sin x$ are both anti-derivatives of $f(x) = \cos x$. We have already resolved this problem when discussing the Mean Value Theorem, but here it is again.

Theorem. *If $F(x)$ and $G(x)$ are anti-derivatives of $f(x)$ on an interval I, then*

$$G(x) = F(x) + c \quad \text{where } c \text{ is some constant.}$$

Proof. Let $H = F - G$. Let $a \in I$ be fixed and let x be any other value in I. By the Mean Value Theorem we see that there exists some ξ between a and x for which

$$H'(\xi) = \frac{H(x) - H(a)}{x - a}$$

However, $H'(\xi) = F'(\xi) - G'(\xi) = f(\xi) - f(\xi) = 0$ for all $\xi \in I$, whence

$$H(x) = H(a)$$

We conclude that $H(x) = c$ is constant. Hence result. ∎

Examples

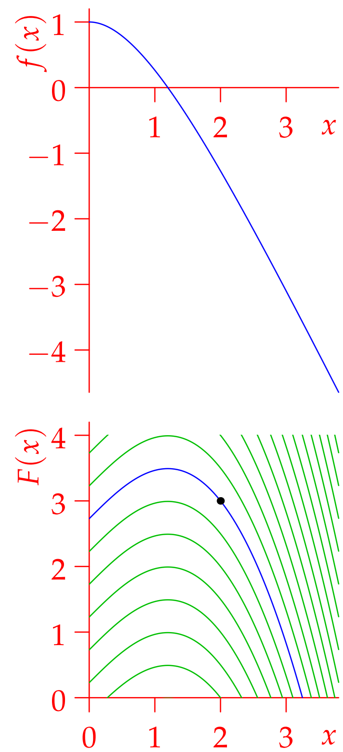
1. The function $f(x) = x + 3x^3 + 29 \sin x$ has domain $\mathbb{R}$ (a single interval), whence all the anti-derivatives of f have the form

   $$F(x) = \frac{1}{2}x^2 + \frac{3}{4}x^4 - 29 \cos x + c$$

   for some constant c.

2. (a) Find all the anti-derivatives of $f(x) = 3 - 2x - 2e^{-x}$.

(b) Find the unique anti-derivative of f which passes through the point $(2, 3)$.



First observe that the domain of f is a single interval $\mathbb{R}$. Appealing to the table, the general anti-derivative of f is

$$F(x) = 3x - x^2 + 2e^{-x} + c$$

where c is any constant.

If the graph of $y = F(x)$ is to pass through the point $(2, 3)$, then we must have

$$3 = F(2) = 6 - 4 + 2e^{-2} + c \implies c = .$$

Therefore

$$F(x) = 3x - x^2 + 2e^{-x} + 1 - 2e^{-2}$$

The function f and several of its anti-derivatives F are drawn, with the desired answer to part (b) in blue.

What if the domain is not an interval? The Theorem says that anti-derivatives differ by a constant for any function defined on an interval. If a function f has a domain that is *not* an interval, then there are more possibilities for anti-derivatives F: in fact we obtain a new arbitrary constant for *every* interval of continuity of f.

Example Find the most general anti-derivative of the function $f(x) = \frac{1}{x}$ defined on the domain $(-\infty, 0) \cup (0, \infty)$.

Looking at the table, we see that $F(x) = \ln|x|$ is an anti-derivative of $f(x)$. Does this mean that *all* anti-derivatives have the form $\ln|x| + c$ for a constant c? The answer is *no*!

- On the interval $(0, \infty)$, all anti-derivatives have the form $F(x) = \ln|x| + c_1 = \ln x + c_1$ for some constant c_1.
- On the interval $(-\infty, 0)$, all anti-derivatives have the form $F(x) = \ln|x| + c_2 = \ln(-x) + c_2$ for some constant c_2.

Since the intervals do not intersect, there is nothing that says that the constants must be the same! The most general anti-derivative is therefore

$$F(x) = \begin{cases} \ln x + c_1 & \text{if } x > 0 \\ \ln(-x) + c_2 & \text{if } x < 0 \end{cases} \qquad \text{where } c_1 \text{ and } c_2 \text{ are constant.}$$

Because of the ulginess of the above, mathematicians will typically still write $F(x) = \ln|x| + c$. It is up to the reader to understand in this case that, for a given anti-derivative, the constant c could be different on *each interval of continuity* of f.

Anti-differentiation and Physics

Suppose that a particle moves under the influence of a force F. The beauty of Newton's Second Law $F = ma$ is that it tells you something about the *second derivative* $a = \frac{d^2s}{dt^2}$ of the position s of the particle. A great deal of Physics is based on solving this *differential equation:* given a force and some initial conditions,[2] find its location at time t. Here we discover the standard formulæ of kinematics using anti-differentiation.

An object of mass m is acted on by a constant force (gravity) of magnitude mg (ms^{-2}). Its initial velocity is v_0 (ms^{-1}) and its initial position is s_0 (m) from a fixed point.

- Newton's Second Law says that $a = \frac{d^2s}{dt^2} = g$.
- Recall that acceleration is the derivative of velocity $v(t)$. Since g is constant, the anti-derivatives of a are of the form $v(t) = gt + c$ where c is constant.[3]
- Since $v(0) = v_0$, we conclude that $c = v_0$ and so

  $$v(t) = gt + v_0$$

- Velocity is the derivative of position $s(t)$. Since g and v_0 are constant, all anti-derivatives of v have the form $s(t) = \frac{1}{2}gt^2 + v_0t + \hat{c}$, where $\hat{c}$ is constant.
- Since $s(0) = s_0$, we conclude that $\hat{c} = s_0$ and so

  $$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$$

What this shows is that the standard high-school kinematics equations for falling objects depend *only* on

- Newton's Second Law.
- A constant force.
- Calculus!

Example An astronaut jumps on the moon ($g = -1.625\,ms^{-2}$) and reaches a maximum height of 3.25 m. How long was the astronaut in the 'air'?

We compute how long it takes the astronaut to fall 3.25 m. Suppose that $t = 0$ at the top of the jump, then $s_0 = 3.25$ and $v_0 = 0$. It follows that

$$s(t) = -0.8125t^2 + 3.25 = 0 \iff t = \sqrt{\frac{3.25}{0.8125}} = \sqrt{4} = 2\,s.$$

It follows that the astronaut was off the ground for a total of 4 s.

[2] Where does the particle start and how fast is it initially travelling?

[3] The domain of all our functions is *time*, which is certainly a single interval!