4.9 Anti-derivatives

Anti-differentiation is exactly what it sounds like: the opposite of differentiation. That is, given a function \( f \), can we find a function \( F \) whose derivative is \( f \).

**Definition.** An anti-derivative of a function \( f \) is a function \( F \) such that \( F'(x) = f(x) \) for all \( x \).

**Examples**

1. \( F(x) = \sin x \) is an anti-derivative of \( f(x) = \cos x \).
2. \( F(x) = (x^2 + 3)^{3/2} \) is an anti-derivative of \( f(x) = 3x\sqrt{x^2 + 3} \).
3. \( F(x) = \frac{x^2 - 3}{\cos x} \) is an anti-derivative of \( f(x) = \frac{2x + (x^2 - 3)\tan x}{\cos x} \).

We can easily check the veracity of these statements by differentiating \( F(x) \). But what if you are asked to find an anti-derivative, not just check something you’ve been given? In general this is a hard problem.

The only method that really exists for explicitly computing anti-derivatives is guess and differentiate! Indeed every famous rule that you’ll study in Integration (substitution, integration by parts, etc.) is merely the result of guessing a general anti-derivative and checking that your guess is correct!

**Anti-derivatives of Common Functions**  Guessing is, of course, easier if you have familiarity with differentiation. With each of the following functions \( f(x) \) you should be able to guess the chosen anti-derivative \( F(x) \) just from what you know about derivatives.

<table>
<thead>
<tr>
<th>Function ( f(x) )</th>
<th>Anti-derivative ( F(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( kf(x) ), ( k ) constant</td>
<td>( kF(x) )</td>
</tr>
<tr>
<td>( f(x) + g(x) )</td>
<td>( F(x) + G(x) )</td>
</tr>
<tr>
<td>( x^n ), ( n \neq -1 )</td>
<td>( \frac{1}{n+1}x^{n+1} )</td>
</tr>
<tr>
<td>( x^{-1} )</td>
<td>( \ln</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( \sin x )</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>( -\cos x )</td>
</tr>
<tr>
<td>( \sec^2 x )</td>
<td>( \tan x )</td>
</tr>
<tr>
<td>( e^{kx} )</td>
<td>( \frac{1}{k}e^{kx} )</td>
</tr>
</tbody>
</table>

**Examples**

1. \( f(x) = 3 + \frac{2}{x} \) has an anti-derivative \( F(x) = 3x + 2\ln|x| \).
2. \( f(x) = 3\sec^2 x - 2e^{3x} \) has an anti-derivative \( F(x) = 3\tan x - \frac{2}{3}e^{3x} \).

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1 Differentiation is often described as easy in the sense that nice functions can usually be differentiated via our familiar rules. Anti-differentiation is hard: very few functions have anti-derivatives that can easily be computed. For example, it is easy to find the derivative of \( f(x) = e^{-x^2} \), namely \( f'(x) = -2xe^{-x^2} \), but can you find an anti-derivative of \( f(x) \)? If, by this, you mean an explicit formula involving simple functions (algebraic, trigonometric, exponential), then the answer is no. The best we can do is to prove that an anti-derivative exists and then estimate it to whatever degree of accuracy we need.
3. Find an anti-derivative of \( g(x) = \cos x + \frac{x^3 + \sqrt{x}}{x^2} \).

The trick is to manipulate \( g(x) \) so that its terms are in the table. We know that \( \sin x \) is an anti-derivative of \( \cos x \), so the challenge is to deal with the second term. Thankfully we can rewrite it as a sum of powers of \( x \):

\[
\frac{x^3 + \sqrt{x}}{x^2} = x + x^{-3/2}
\]

which can now be anti-differentiated. It follows that a suitable anti-derivative of \( g(x) \) is

\[
G(x) = \sin x + \frac{1}{2} x^2 - 2x^{-1/2}
\]

**How many anti-derivatives does a function have?**

The definition of anti-derivative does nothing more than providing an alternative way of referring to a pair of functions \( f \) and \( F \) satisfying the equation \( f(x) = F'(x) \). That is, the following statements mean exactly the same thing.

- \( f(x) \) is the derivative of \( F(x) \).
- \( F(x) \) is an anti-derivative of \( f(x) \).

The crucial observation is the difference between the articles *the* and *an*: a given function has at most one derivative, but (potentially) many anti-derivatives. Indeed, \( F(x) = \sin x \) and \( G(x) = 17 + \sin x \) are both anti-derivatives of \( f(x) = \cos x \). We have already resolved this problem when discussing the Mean Value Theorem, but here it is again.

**Theorem.** If \( F(x) \) and \( G(x) \) are anti-derivatives of \( f(x) \) on an interval \( I \), then

\[
G(x) = F(x) + c \quad \text{where } c \text{ is some constant.}
\]

**Proof.** Let \( H = F - G \). Let \( a \in I \) be fixed and let \( x \) be any other value in \( I \). By the Mean Value Theorem we see that there exists some \( \xi \) between \( a \) and \( x \) for which

\[
H'(\xi) = \frac{H(x) - H(a)}{x - a}
\]

However, \( H'(\xi) = F'(\xi) - G'(\xi) = f(\xi) - f(\xi) = 0 \) for all \( \xi \in I \), whence

\[
H(x) = H(a)
\]

We conclude that \( H(x) = c \) is constant. Hence result.

**Examples**

1. The function \( f(x) = x + 3x^3 + 29 \sin x \) has domain \( \mathbb{R} \) (a single interval), whence all the anti-derivatives of \( f \) have the form

\[
F(x) = \frac{1}{2} x^2 + \frac{3}{4} x^4 - 29 \cos x + c
\]

for some constant \( c \).
2. (a) Find all the anti-derivatives of \( f(x) = 3 - 2x - 2e^{-x} \).

(b) Find the unique anti-derivative of \( f \) which passes through the point \((2,3)\).

First observe that the domain of \( f \) is a single interval \( \mathbb{R} \). Appealing to the table, the general anti-derivative of \( f \) is

\[
F(x) = 3x - x^2 + 2e^{-x} + c
\]

where \( c \) is any constant.

If the graph of \( y = F(x) \) is to pass through the point \((2,3)\), then we must have

\[
3 = F(2) = 6 - 4 + 2e^{-2} + c \implies c = .
\]

Therefore

\[
F(x) = 3x - x^2 + 2e^{-x} + 1 - 2e^{-2}
\]

The function \( f \) and several of its anti-derivatives \( F \) are drawn, with the desired answer to part (b) in blue.

**What if the domain is not an interval?** The Theorem says that anti-derivatives differ by a constant for any function defined on an interval. If a function \( f \) has a domain that is *not* an interval, then there are more possibilities for anti-derivatives \( F \): in fact we obtain a new arbitrary constant for every interval of continuity of \( f \).

**Example** Find the most general anti-derivative of the function \( f(x) = \frac{1}{x} \) defined on the domain \(( -\infty, 0) \cup (0, \infty) \).

Looking at the table, we see that \( F(x) = \ln|x| \) is an anti-derivative of \( f(x) \). Does this mean that *all* anti-derivatives have the form \( \ln\ |x| + c \) for a constant \( c \)? The answer is *no!*

- On the interval \((0, \infty)\), all anti-derivatives have the form \( F(x) = \ln |x| + c_1 = \ln x + c_1 \) for some constant \( c_1 \).

- On the interval \(( -\infty, 0)\), all anti-derivatives have the form \( F(x) = \ln |x| + c_2 = \ln(-x) + c_2 \) for some constant \( c_2 \).

Since the intervals do not intersect, there is nothing that says that the constants must be the same! The most general anti-derivative is therefore

\[
F(x) = \begin{cases} 
\ln x + c_1 & \text{if } x > 0 \\
\ln(-x) + c_2 & \text{if } x < 0 
\end{cases}
\]

where \( c_1 \) and \( c_2 \) are constant.

Because of the ugliness of the above, mathematicians will typically still write \( F(x) = \ln |x| + c \). It is up to the reader to understand in this case that, for a given anti-derivative, the constant \( c \) could be different on each interval of continuity of \( f \).
Anti-differentiation and Physics

Suppose that a particle moves under the influence of a force $F$. The beauty of Newton’s Second Law $F = ma$ is that it tells you something about the second derivative $a = \frac{d^2s}{dt^2}$ of the position $s$ of the particle. A great deal of Physics is based on solving this differential equation: given a force and some initial conditions, find its location at time $t$. Here we discover the standard formulæ of kinematics using anti-differentiation.

An object of mass $m$ is acted on by a constant force (gravity) of magnitude $mg$ (ms$^{-2}$). Its initial velocity is $v_0$ (ms$^{-1}$) and its initial position is $s_0$ (m) from a fixed point.

- Newton’s Second Law says that $a = \frac{d^2s}{dt^2} = g$.
- Recall that acceleration is the derivative of velocity $v(t)$. Since $g$ is constant, the anti-derivatives of $a$ are of the form $v(t) = gt + c$ where $c$ is constant.$^3$
- Since $v(0) = v_0$, we conclude that $c = v_0$ and so $v(t) = gt + v_0$

- Velocity is the derivative of position $s(t)$. Since $g$ and $v_0$ are constant, all anti-derivatives of $v$ have the form $s(t) = \frac{1}{2}gt^2 + v_0t + \hat{c}$, where $\hat{c}$ is constant.
- Since $s(0) = s_0$, we conclude that $\hat{c} = s_0$ and so $s(t) = \frac{1}{2}gt^2 + v_0t + s_0$

What this shows is that the standard high-school kinematics equations for falling objects depend only on

- Newton’s Second Law.
- A constant force.
- Calculus!

Example  
An astronaut jumps on the moon ($g = -1.625$ ms$^{-2}$) and reaches a maximum height of 3.25 m. How long was the astronaut in the ‘air’?

We compute how long it takes the astronaut to fall 3.25 m. Suppose that $t = 0$ at the top of the jump, then $s_0 = 3.25$ and $v_0 = 0$. It follows that

$$s(t) = -0.8125t^2 + 3.25 = 0 \iff t = \sqrt{\frac{3.25}{0.8125}} = \sqrt{4} = 2 \text{ s}.$$  

It follows that the astronaut was off the ground for a total of 4 s.

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$^2$Where does the particle start and how fast is it initially travelling?

$^3$The domain of all our functions is time, which is certainly a single interval!