11.2 Infinite Series

Once we have a sequence of numbers, the next thing to do is to sum them up. Given a sequence $(a_n)_{n=1}^{\infty}$ be a sequence: can we give a sensible meaning to the following expression?

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

While summing infinitely many terms may seem strange, what isn't strange is to sum *finitely* many terms. We therefore make a definition.

Definition. Let $(a_n)_{n=1}^{\infty}$ be a sequence. The *nth partial sum* of the sequence is the value

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Example Suppose that $a_n = \frac{1}{2^n}$ so that

$$(a_n) = \left(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots\right)$$

We compute the first few partial sums:

$$\begin{aligned} s_1 &= a_1 = \frac{1}{2} \\ s_2 &= a_1 + a_2 = \frac{3}{4} \\ s_3 &= a_1 + a_2 + a_3 = \frac{7}{8} \\ s_4 &= a_1 + a_2 + a_3 + a_4 = \frac{15}{16} \end{aligned}$$

What we are constructing is a new sequence $(s_n)_{n=1}^{\infty}$, the *sequence of partial sums.* It certainly looks like we have a formula for the nth term of this sequence

$$s_n = 1 - \frac{1}{2^n}$$

Moreover, the limit of the sequence is $\lim_{n\to\infty} s_n = 1$. It seems reasonable to define this to be the infinite sum of the original sequence, and we write

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$$

In general, we have the following:

Definition. Let $(a_n)_{n=1}^{\infty}$ be a sequence. The *infinite series* $\sum_{n=1}^{\infty} a_n$ is the limit of the sequence of partial sums, if it exists. That is

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

We say that the series *converges* (or diverges) if and only if the above limit does.

'Zeroth' terms and notation All of our previous discussion could have been performed with a sequence starting with a differently indexed term, i.e. $(a_n)_{n=m}^{\infty} = (a_m, a_{m+1}, a_{m+2}, \ldots)$. The most common choices are $m = 0$ or 1. The term a_0 is often referred to as the *zeroth term* of the sequence.
If we know what the initial term of a sequence is, or if a result does not depend on the initial term, then it is common to omit the limits entirely and simply denote the series by $\sum a_n$.

Series Laws

Series behave exactly like finite sums. Therefore, if the series $\sum a_n$ and $\sum b_n$ converge, and if c is a constant, we have

1. $\sum ca_n = c \sum a_n$ (compare the finite sum $ca_1 + ca_2 = c(a_1 + a_2)$)
2. $\sum a_n + \sum b_n = \sum (a_n + b_n)$ (compare $(a_1 + a_2) + (b_1 + b_2) = (a_1 + b_1) + (a_2 + b_2)$)

These expressions are akin to the limit laws. By contrast with limits, series *do not* obey laws regarding products and division. Thus

$$\left(\sum a_n\right)\left(\sum b_n\right) \neq \sum a_n b_n \qquad \text{and} \qquad \frac{\sum a_n}{\sum b_n} \neq \sum \frac{a_n}{b_n}$$

Again, if you imagine what these would mean for finite sums,[1] there is no reason to expect equality!

Geometric Series

Perhaps the most important family of infinite series are those obtained by summing a *geometric sequence:* that is a sequence of the form $a_n = ar^n$ where a and r are constants. The motivating example above is such a sequence, with $r = \frac{1}{2}$. We can deal with these in general. There is something of a convention regarding the first term of a geometric series, so it is worth making a new definition.

Definition. A *geometric series* is an infinite series of the form $\sum_{n=0}^{\infty} ar^n$ for some constants $a \neq 0$ and r. In particular, the sequence (a_n) starts with the *zeroth term* $a_0 = ar^0 = a$, where we follow the convention that $r^0 = 1$.

Theorem. *The geometric series with nth term $a_n = r^n$ converges if and only if $-1 < r < 1$, in which case*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Proof. Suppose first that $r = 1$. In this case the nth partial sum is simply

$$s_n = \underbrace{1 + 1 + \cdots + 1 + 1}_{n \text{ times}} = n$$

Clearly the sequence (s_n) diverges to $\pm\infty$, depending on the sign of a, whence $\sum a_n$ diverges.
Now suppose that $r \neq 1$. Consider the nth partial sum.

$$s_n = 1 + {\color{red}r} + {\color{blue}r^2} + \cdots + r^{n-1} + {\color{green}r^n}$$

[1]Clearly $(a_1 + a_2)(b_1 + b_2) \neq a_1b_1 + a_2b_2$ and $\dfrac{a_1 + a_2}{b_1 + b_2} \neq \dfrac{a_1}{b_1} + \dfrac{a_2}{b_2}$

Multiply this by r.

$$rs_n = r + r^2 + r^3 + \cdots + r^n + r^{n+1}$$

Now subtract one line from the other, noticing how almost all the terms come in cancelling pairs:

$$(1-r)s_n = 1 - r^{n+1}$$

We have therefore obtained an nth term formula for the sequence of partial sums

$$s_n = \frac{1-r^{n+1}}{1-r}$$

As we saw in the previous section, $\lim_{n\to\infty} r^{n+1}$ converges if and only if $-1 < r \le 1$. Since $r \neq 1$ in this case, we also see that this limit is zero, which gives the result.

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Examples

1. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3.$

2. If the summation does not start with a zeroth term, then it is a good idea to re-index so that it does. In what follows, we let $m = n - 1$.

$$\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^n = -\frac{3}{4}\sum_{m=0}^{\infty} \left(-\frac{3}{4}\right)^m = -\frac{3}{4} \cdot \frac{1}{1-(-3/4)} = -\frac{3}{7}$$

   If the above feels too fast, try writing out the first few terms of the series: e.g.

$$\begin{aligned}\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^n &= -\frac{3}{4} + \left(-\frac{3}{4}\right)^2 + \left(-\frac{3}{4}\right)^3 + \left(-\frac{3}{4}\right)^4 + \cdots \\ &= -\frac{3}{4}\left[1 - \frac{3}{4} + \left(-\frac{3}{4}\right)^2 + \left(-\frac{3}{4}\right)^3 + \cdots\right] \\ &= -\frac{3}{4}\sum_{m=0}^{\infty} \left(-\frac{3}{4}\right)^m\end{aligned}$$

   If you think about the initial term you shouldn't go wrong!

3. Sometimes a little more work with exponential laws is required in order to view a series as geometric.

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{3^{2n+1}}{2^{4n+3}} &= \frac{3}{8}\sum_{n=2}^{\infty} \frac{(3^2)^n}{(2^4)^n} = \frac{3}{8}\sum_{n=2}^{\infty} \left(\frac{9}{16}\right)^n = \frac{3}{8}\left(\frac{9}{16}\right)^2 \sum_{m=0}^{\infty} \left(\frac{9}{16}\right)^m \\ &= \frac{3 \cdot 9^2}{8 \cdot 16^2} \cdot \frac{1}{1-\frac{9}{16}} = \frac{3 \cdot 81}{8 \cdot 16 \cdot 7} = \frac{243}{896}\end{aligned}$$

4. When a geometric series diverges it is very easy to spot. For instance

$$\sum_{n=5}^{\infty} \frac{3^{2n+1}}{2^{3n-4}} = \frac{3}{2^{-4}}\sum_{n=5}^{\infty} \left(\frac{9}{8}\right)^n$$

   diverges to infinity since $\frac{9}{8} > 1$.

Converting a repeating decimal into a fraction Geometric series are also useful for understanding repeating decimals. For example,

$$\begin{aligned}3.15151515\ldots &= 3 + \frac{15}{100} + \frac{15}{100^2} + \frac{!5}{100^3} + \cdots \\ &= 3 + \frac{15}{100}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n = 3 + \frac{15}{100}\cdot\frac{1}{1-\frac{1}{100}} \\ &= 3 + \frac{5}{33} = \frac{104}{33}\end{aligned}$$

Indeed it is a theorem that every decimal which eventually has a repeating pattern must be a rational number. Try ton convince yourself using geometric series that

$$2.125271271271271\ldots = \frac{1061573}{499500}$$

In particular this shows that the decimal representation of an irrational number such as $\sqrt{2}$ or π will never have a repeating block of digits!

Telescoping Series Beyond geometric series, there are very few series that we can compute exactly. One such family are known as telescoping series, and the idea for how to deal with them is analogous to the partial fractions method for integration. Here is an example: to compute the value of the infinite series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

we first consider the nth partial sum:

$$\begin{aligned}s_n &= \sum_{i=1}^{n}\frac{1}{i(i+1)} = \sum_{i=1}^{n}\frac{1}{i} - \frac{1}{i+1} && \text{(partial fractions decomposition)} \\ &= \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1}-\frac{1}{n}\right) + \left(\frac{1}{n}-\frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} && \text{(cancel terms in pairs)}\end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = 1$$

For a more complicated example, consider

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} = \sum_{n=2}^{\infty}\frac{1}{(n-1)(n+1)} = \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

with nth partial sum satisfying

$$\begin{aligned}
2s_n &= \left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\cdots \\
&\qquad\qquad \cdots+\left(\frac{1}{n-3}-\frac{1}{n-1}\right)+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right) \\
&= 1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}
\end{aligned}$$

from which

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} = \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)=\frac{3}{4}$$

Showing Divergence

It is often easier to prove that a series diverges than to prove convergence, as the following result shows.

Theorem (nth-term/divergence test)**.** *If a_n does not converge to zero, then $\sum a_n$ does not converge.*

Proof. We prove the contrapositive.[2] Assume that $\sum a_n$ converges to s. Then

$$s = \lim_{n\to\infty} s_n$$

where (s_n) is the sequence of partial sums. But then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = s - s = 0$$

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Example $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$ diverges since

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1}\neq 0$$

Non-example The nth-term test only works in one direction! As we shall see in the next section, the *harmonic series* $\sum\frac{1}{n}$ *diverges*, even though the sequence $\frac{1}{n}$ converges to zero.

Suggested problems

1. Find the sum of each of the following series:

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{3}{2^n}$

   (b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(n+1)}$

[2] If $\sum a_n$ converges then (a_n) converges to zero. This statement is logically equivalent to that in the Theorem.

2. Express the decimal $3.4545\overline{45}$ as a fraction.

3. Suppose you borrow \$20,000 for a new car at a monthly interest rate[3] of 0.5%. Suppose you make payments of \$600 per month, paid at the end of each month.

   (a) Let a_n be the amount you owe at the start of the nth month. Show that $a_{n+1} = 1.0075a_n - 600$.

   (b) Let $b_n = a_n - 80000$. Find a recurrence relation for b_n and solve it.

   (c) After how many months will the loan balance be zero?

[3] I.e. at the end of the first month you owe an extra \$100 interest.