11.7 Strategy for Testing Series

Most questions related to series in this course are a rephrasing of the following:

Given a series $\sum a_n$, decide if the series converges or diverges: if it converges, decide whether the series converges absolutely or conditionally.

The challenge is to be able to quickly choose the most useful test, or combination of tests, to efficiently answer the question. Sometimes several tests will apply. In large part, making the right choice is something that comes with experience: here are some suggestions for the order in which to consider each test, where the order is chosen to reflect the balance between applicability and ease of use.

1 *nth term/divergence test*

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum a_n \quad \text{divergent}$$

The nth term test comes first because it is very easy and quick to use. Spend 10 seconds considering whether it applies; if it does, the question is over very quickly. The weakness in the nth term test is that it can *only* show that a series $\sum a_n$ *diverges*. The test can *never* show that a series *converges.*

2 *Standard series, or a simple combination?* You should know:

p-series: $\sum \frac{1}{n^p}$ converges if and only if $p > 1$

Geometric series: $\sum r^n$ converges if and only if $-1 < r < 1$

3 *Test for Absolute Convergence* You should consider the following tests for the series $\sum |a_n|$.

(a) (Limit) Comparison Test Compare your series with something similar, but simpler. A simplistic summary of the tests is as follows:

Comparison Test $|a_n| \leq |b_n| \implies \sum |a_n| \leq \sum |b_n|$

Limit Comparison Test If $\lim\limits_{n\to\infty} \frac{|a_n|}{|b_n|} \in (0, \infty)$, then $\sum |a_n|$ and $\sum |b_n|$ have the same convergence status.

(b) Ratio Test Consider $L = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Absolute convergence if $L < 1$, divergence if $L > 1$. Useful if series contains factorials and/or exponentials.

(c) Root Test Consider $L = \lim\limits_{n\to\infty} |a_n|^{1/n}$. Absolute convergence if $L < 1$, divergence if $L > 1$. Useful if the nth term of the series is the nth power of something simple.

(d) Integral Test If $|a_n| = f(n)$ is decreasing on $[N, \infty)$ then

$$\sum_{n=N}^{\infty} |a_n| \text{ converges} \iff \int_N^\infty f(x)\,\mathrm{d}x \text{ converges}$$

The integral test is late in the list because it rarely applies since integration is difficult!

7 *Alternating Series Test* If a_n alternates between positive and negative, and $|a_n|$ is decreasing to zero, then $\sum a_n$ converges. This is the *only* test that can show *conditional convergence.* Useless if any of the above tests show absolute convergence.

Values of convergent series It is also possible to be asked to find the explicit value of a convergent series. The only series for which you should know how to do this (at this stage) are:

Geometric Series with $|r| < 1$ $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ or more generally $\sum_{n=N}^{\infty} r^n = \frac{r^N}{1-r}$ which can be remembered as

$$\sum r^n = \frac{\text{initial term}}{1 - \text{ratio of successive elements}}$$

Telescoping Series These are special series which may be written in such a fashion that most terms of the series cancel. For example

$$\sum_{k=2}^{n} \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+1}}$$

from which

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{2}}$$

You may need to use some algebraic trickery to put a series in telescoping form.

Examples For each of these, you should be able to run through the checklist in your head until you reach something for which a calculation is merited. In parentheses $\Big(\Big)$ is the a typical thought process: you don;t need to write down *anything* contained therein.

1. $\sum_{n=2}^{\infty} \frac{e^{1/n}}{n^2}$

   $\Big(\lim_{n\to\infty} \frac{e^{1/n}}{n^2} = 0$, so the nth term test does not apply.
   Not a p-series or geometric series.
   However $\lim_{n\to\infty} e^{1/n} = 1$, whence we consider a comparison with $\sum \frac{1}{n^2}$.$\Big)$

   We compare with the convergent p-series $\sum_{n=2}^{\infty} \frac{1}{n^2}$.

   $$\lim_{n\to\infty} \frac{e^{1/n}/n^2}{1/n^2} = \lim_{n\to\infty} e^{1/n} = 1$$

   By the limit comparison test, that $\sum_{n=2}^{\infty} \frac{e^{1/n}}{n^2}$ converges.

2. $\sum_{n=3}^{\infty} \cos(n^2)$ is divergent by the nth term test, since $\cos(n^2)$ is divergent (hence does not converge to zero).

3. $\displaystyle\sum_{j=0}^{\infty} \frac{(-1)^j \sqrt{j}}{j+5}$

$\Big($Sequence converges to zero, so nth term test does not apply.
Not a standard series, or combination thereof.
When j is large, $\sum |a_j|$ looks like the divergent p-series $\sum j^{-1/2}$.$\Big)$

We compare with the divergent p-series $\sum\limits_{n=1}^{\infty} \frac{1}{j^{1/2}}$. We have

$$\lim_{j\to\infty} \frac{\sqrt{j}/(j+5)}{1/j^{1/2}} = \frac{j}{j+5} = 1$$

whence $\sum\limits_{j=1}^{\infty} \frac{\sqrt{j}}{j+5}$ diverges. Certainly the original series (summed from $j = 0$) does not converge absolutely.[1]

$\Big($Remains to see if series conditionally converges.
Only test that can do this is the alternating series test.
Must check that sequence $\frac{\sqrt{j}}{j+5}$ decreases to zero.$\Big)$

Let $f(x) = \frac{\sqrt{x}}{x+5}$. Then

$$f'(x) = \frac{\frac{1}{2}x^{-1/2}(x+5) - x^{1/2}}{(x+5)^2} = \frac{x+5-2x}{2x^{1/2}(x+5)^2} = \frac{5-x}{2x^{1/2}(x+5)^2}$$

which is negative if $x > 5$. Therefore $\left(\frac{\sqrt{j}}{j+5}\right)_{j=5}^{\infty}$ is decreasing. Clearly the limit of this sequence is zero. By the alternating series test, the series $\sum\limits_{j=5}^{\infty} \frac{(-1)^j\sqrt{j}}{j+5}$ converges. It follows that the original series $\sum\limits_{j=0}^{\infty} \frac{(-1)^j\sqrt{j}}{j+5}$ converges conditionally.

4. $\displaystyle\sum_{k=2}^{\infty} \frac{k^2}{e^{k^2}}$

$\Big($Sequence converges to zero (exponential grows faster than k^2) so nth term test says nothing.
Comparison not obvious.
Contains exponential, so try ratio test.$\Big)$

Writing $a_k = \frac{k^2}{e^{k^2}}$, we have

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{(k+1)^2 e^{k^2}}{k^2 e^{(k+1)^2}} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^2 \cdot \frac{1}{e^{(k+1)^2-k^2}} = \lim_{k\to\infty} \frac{1}{e^{2k+1}} = 0$$

Since $0 < 1$, the ratio test says that the series $\sum\limits_{k=2}^{\infty} \frac{k^2}{e^{k^2}}$ is (absolutely) convergent.

[1] Note that the ratio test would be useless here since there are no exponentials or factorials. Try it...

5. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n - 3n^2}{6^n + n}$

$\left(\text{When } n \text{ is large this looks like the convergent geometric series } \sum \frac{4^n}{6^n}\right)$

We compare with the convergent geometric series $\sum \frac{4^n}{6^n}$. Certainly the series is a sum of positive terms.[2] Indeed, if $n \geq 1$, we have

$$\frac{4^n - 3n^2}{6^n + n} \leq \frac{4^n}{6^n + n} \leq \frac{4^n}{6^n}$$

By the comparison test we conclude that

$$\sum_{n=1}^{\infty} \frac{4^n - 3n^2}{6^n + n} \leq \sum_{n=1}^{\infty} \frac{4^n}{6^n}$$

whence the former converges.

6. Compare the convergence statuses of $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-n}{2n+1}\right)^{5n}$, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-n}{n+1}\right)^{5n}$ and $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

$\left(\text{Since all series involve } n\text{th powers, the root test seems the obvious candidate.}\right)$

Note that

$$\lim_{n\to\infty} \left| \left(\frac{-n}{2n+1}\right)^{5n} \right|^{1/n} = \lim_{n\to\infty} \left(\frac{n}{2n+1}\right)^5 = \frac{1}{2^5}$$

Since $\frac{1}{2^5} < 1$ the root test says that the first series converges absolutely.

(It should be obvious now that applying the root test to the second series will give the inconclusive limit 1, so we need something else. Does the *sequence* go to zero?)

Observe that

$$\lim_{n\to\infty} \left| \left(\frac{-n}{n+1}\right)^{5n} \right| = \lim_{n\to\infty} \left(\frac{1}{(n+1)/n}\right)^{5n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-5n} = e^{-5}$$

In particular, $\left(\frac{-n}{n+1}\right)^{5n}$ does not converge to zero: by the nth term test, the second series diverges. The nth term of the final series clearly has absolute value larger than that of the second series. Clearly this sequence does not converge to zero either, and the series consequently diverges.

[2] If you doubt this, consider the first few numerators and note that the exponential 4^n increases much quicker than $3n^2$.