11.8 Power Series

Power series are essentially polynomials of infinite degree. For example, recall that the geometric series $\sum_{n=0}^{\infty} r^n$ converges to $\frac{1}{1-r}$ if $|r| < 1$. We could imagine the series as a *function* of the variable $x = r$ and write

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Given that the series converges if and only if $-1 < x < 1$, we say that the power series/infinite polynomial *represents the function* $\frac{1}{1-x}$ on the interval $(-1, 1)$.
Power series have many of the advantages of polynomials in that they are easy to add and to differentiate/integrate. The history of power series is long: in the days before concepts such as sine and cosine were thought of as functions, Issac Newton did much of his work using power series. The delicate issues of convergence were ironed out some decades after Newton's time.

Definition. The *power series* with center a and coefficients c_n is the expression[1]

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

For each fixed value of x, a power series is an infinite series. It therefore converges or diverges. It is possible for a power series to converge for some values of x and diverge for others. We can think of the set of values x for which a power series converges as the domain of a *function*:

$$p(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

For example, if $p(x) = \sum\limits_{n=0}^{\infty} x^n$ then $p(x) = \dfrac{1}{1-x}$ with domain $(-1, 1)$.
Certainly a power series always converges at its center where $p(a) = c_0$. It will be seen that a power series need not converge at any other values of x.

Finding the domain of a power series

The ratio test is often very effective for finding where a power series converges. For example, suppose that

$$p(x) = \sum_{n=0}^{\infty} \frac{n-1}{2^n} x^n = -1 + \frac{1}{4}x^2 + \frac{1}{4}x^3 + \frac{3}{16}x^4 + \frac{1}{8}x^5 + \cdots$$

If we imagine this as an infinite series of the form $\sum a_n$, we can apply the ratio test: simply pretend that x is a constant until we are done.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n\,|x|^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(n-1)\,|x|^n} = \frac{n}{2(n-1)}\,|x| \underset{n\to\infty}{\longrightarrow} \frac{|x|}{2} \quad \begin{cases} < 1 \iff |x| < 2 \\ > 1 \iff |x| > 2 \end{cases}$$

[1] Throughout we use the convention that $(x-a)^0 = 1$, even if $x = a$. We will similarly use the convention that $0! = 1$. A power series is always considered to start with a *zeroth term* c_0. It is of course possible that $c_0 = 0$, but we abstractly always work with n counting from zero.

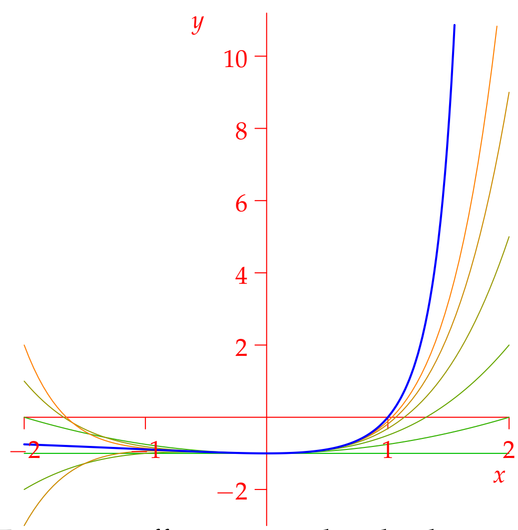
The outcome of the ratio test depends on the value of x. If $|x| < 2$, then the ratio test says that the power series $p(x)$ *converges absolutely*. If $|x| > 2$ then the power series *diverges*.
It remains to consider what happens when $x = \pm 2$.

If $x = 2$: The series becomes $\sum_{n=0}^{\infty} (n-1)$ which diverges to infinity.

If $x = -2$: The series becomes $\sum_{n=0}^{\infty} (-1)^n (n-1)$ which diverges by oscillation.

Therefore the power series converges absolutely when $|x| < 2$, and diverges when $|x| \geq 2$. With a little calculus (later) we can see that

$$p(x) = \sum_{n=0}^{\infty} \frac{n-1}{2^n} x^n = \frac{4x-4}{(2-x)^2} \text{ with domain } (-2, 2)$$



What is the point of this exercise? One of the most common applications of power series is to truncate the series after a few terms to obtain a polynomial. This will (hopefully) be a good approximation to the power series function, and will be *much* easier to work with.

In the picture, the curve $y = p(x)$ is drawn in blue, while the other curves are polynomials obtained from the power series by taking only the first few terms (more orange = more terms). It certainly seems that the more terms we take, the better the approximation gets.

The detail of some of this constitutes the rest of the course. To start us off, we notice that the domain of the power series above is an *interval*. In fact this is always the case.

Theorem. *Given a power series $p(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, exactly one of the following is true:*

1. *The series converges* only *when $x = a$*
2. *The series converges* for all $x \in \mathbb{R}$
3. *There is some value $R > 0$ such that series converges absolutely whenever $|x - a| < R$ and diverges whenever $|x - a| > R$*

Definition. We call R the *radius of convergence*. It is a convention to take $R = 0$ in case 1 and $R = \infty$ in case 2 of the Theorem.
The set of values x for which the series converges (the domain of the power series) is termed the *interval of convergence.*

Note that a power series could converge or diverge when $x = a \pm R$. The interval of convergence is therefore exactly one of the following:

$$\{a\}, \quad (-\infty, \infty), \quad [a-R, a+R], \quad (a-R, a+R], \quad [a-R, a+R), \quad (a-R, a+R)$$

We have to test the endpoints of the interval separately to identify the interval of convergence.

While a complete proof of the Theorem is too difficult for this course, we can prove a useful result which often allows us to easily compute the radius of convergence.

Theorem (Ratio Test for Power Series)**.** *The radius of convergence R of a power series $\sum c_n(x-a)^n$ satisfies*

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

if the limit exists or is ∞.

Be careful! The computed limit is *upside down* relative to that in the ratio test!

Proof. Assume that the limit $L = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$ exists, and apply the ratio test to the power series $\sum c_n(x-a)^n$. Then

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| |x-a| = L\,|x-a|$$

By the ratio test, if x is such that right hand side is less than 1, then the power series converges absolutely. If the right hand side is greater than 1, then the power series diverges. There are three cases:

If $L = 0$: Then the series converges for all x, whence $R = \infty$.

If $0 < L < \infty$: Then the series converges when $|x-a| < \frac{1}{L}$ and diverges when $|x-a| > \frac{1}{L}$. By the Theorem, $\frac{1}{L}$ is the radius of convergence.

If $L = \infty$: Then the left hand side of the above converges if and only if $x = a$, whence $R = 0$. ∎

Examples

1. Let $p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Applying the ratio test, we have

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1/n!}{1/(n+1)!} \right| = \lim_{n\to\infty} n+1 = \infty$$

   It follows that the power seres $p(x)$ converges absolutely for all $x \in \mathbb{R}$.

2. Repeating for $p(x) = \sum_{n=0}^{\infty} \frac{n!}{(n+3)^2}(x-4)^n$, we obtain

$$R = \lim_{n\to\infty} \left| \frac{n!(n+4)^2}{(n+1)^2(n+1)!} \right| = \lim_{n\to\infty} \frac{(n+4)^2}{(n+1)^3} = 0$$

   The power series only converges when $x = 4$.

3. $p(x) = \sum_{n=0}^{\infty} \frac{(-2)^n}{2n+1}(x-3)^n$ has

$$R = \lim_{n\to\infty} \left| \frac{(-2)^n(2n+3)}{(2n+1)(-2)^{n+1}} \right| = \lim_{n\to\infty} \frac{2n+3}{2(2n+1)} = \frac{1}{2}$$

Therefore the power series converges absolutely whenever $|x-3| < \frac{1}{2}$ and diverges whenever $|x+3| > \frac{1}{2}$. We still need to test when $|x-3| = \frac{1}{2}$. There are two cases:

If $x - 3 = \frac{1}{2}$*:* then $p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ which converges (conditionally) by the alternating series test.

If $x - 3 = -\frac{1}{2}$*:* then $p(x) = \sum_{n=0}^{\infty} \frac{1}{2n+1}$ which diverges by comparison with the harmonic series $\sum \frac{1}{n}$.

We conclude that the series converges if and only if

$$-\frac{1}{2} < x - 3 \le \frac{1}{2} \iff \frac{5}{2} < x \le \frac{7}{2}$$

Suggested problems

1. Find the radius of convergence and the interval of convergence:

   (a) $\sum_{n=0}^{\infty} \frac{1}{3n+1} x^n$

   (b) $\sum_{n=0}^{\infty} (n+1)2^n x^n$

   (c) $\sum_{n=0}^{\infty} \frac{1}{3^n(n^2+2)} x^n$

2. Find the radius of convergence and the interval of convergence:

   (a) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}(2x+1)^n$

   (b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{8^n \cdot (3n+2)} x^{3n+2}$

3. (a) The *Bessel function of order zero* has the formula

   $$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

   Find its interval of convergence.

   (b) Suppose that the interval of convergence of the power series $\sum c_n(x-1)^n$ is $(-1, 3)$. For what values of x does the expression $\sum \frac{c_n(x^2-3)^n}{3^n}$ converge?