11.9 Representations of Functions as Power Series

As we saw before, one of the main ideas of power series is in trying to represent functions in a different way. The approach really becomes useful when there is no other good way of representing a function. Everything in this section follows from the important fact about geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if and only if} \quad -1 < x < 1$$

Example Given the geometric series formula above, we can replace x with something more complicated, as long as we think about the domain carefully. For instance

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n} = 1 - x^3 + x^6 - x^9 + \cdots$$

This equality holds if and only if $-1 < -x^3 < 1 \iff -1 < x < 1$.

The real benefit is that we can integrate and differentiate this expression, provided x stays between ± 1. Therefore

$$\begin{aligned}\int \frac{1}{1+x^3}\,dx &= \sum_{n=0}^{\infty} (-1)^n \int x^{3n}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} x^{3n+1} \\ &= C + x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \cdots\end{aligned}$$

again, provided that $-1 < x < 1$. This integral could have been computed using partial fractions methods, but it is very tricky and slow: a power series representation is much easier. The following theorem formalizes this *term-by-term* integration and differentiation.

Theorem. *If* $p(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ *has radius of convergence* R*, then*

$$p'(x) = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1} = \sum_{n=0}^{\infty} c_{n+1}(n+1)(x-a)^n$$

$$\int p(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} = C + \sum_{n=1}^{\infty} \frac{c_{n-1}}{n}(x-a)^n$$

Moreover, both have radius of convergence R*.*

The integral and derivative don't necessarily have the same *interval* of convergence, just radius.[1]

Example Find the interval of convergence of $p(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 3^n} x^n$, and its integral and derivative.

Applying the ratio test for power series, we have

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 3^{n+1}}{n^2 3^n} \right| = 3$$

$p(x)$, $\int p(x)\,dx$ and $p'(x)$ therefore all converge absolutely when $|x| < 3$ and diverge when $|x| > 3$. It remains to check the endpoints of the intervals of convergence.

[1] The best we can say is that if $p(x)$ converges *absolutely* at an endpoint of the interval, then so does its integral.

- $p(x) = \sum\limits_{n=1}^{\infty} \frac{1}{n^2 3^n} x^n$: At $x = 3$ we have the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ which converges. At $x = -3$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which also converges. Therefore $p(x)$ has interval of convergence $[-3, 3]$.

- $\int p(x)\,\mathrm{d}x = C + \sum\limits_{n=1}^{\infty} \frac{1}{(n+1)n^2 3^n} x^{n+1}$: At $x = 3$ we have the series $C + \sum\limits_{n=1}^{\infty} \frac{3}{(n+1)n^2}$ which converges by comparison with $\sum \frac{1}{n^3}$. Similarly $\int p(x)\,\mathrm{d}x$ converges at $x = -3$ and therefore has interval of convergence $[-3, 3]$.

- $p'(x) = \sum\limits_{n=1}^{\infty} \frac{1}{n 3^n} x^{n-1}$: At $x = 3$ we have the series $\sum\limits_{n=1}^{\infty} \frac{1}{3n}$ which diverges, while at $x = -3$ we have $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{3n}$ which converges. Therefore $p'(x)$ has interval of convergence $[-3, 3)$.

We can also integrate and differentiate well-understood series term by term.

1. The geometric series $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$. Therefore

$$\ln|1-x| = \int_0^x \frac{1}{1-t}\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} x^n$$

   Replacing x with x^3, it follows that

$$\ln\left|1-x^3\right| = \sum_{n=1}^{\infty} \frac{1}{n} x^{3n}$$

   after which we could integrate again:

$$\int \ln\left|1-x^3\right|\,\mathrm{d}x = C + \sum_{n=1}^{\infty} \frac{1}{n(3n+1)} x^{3n+1}$$

   Since the original series has radius of convergence $R = 1$, so do all of the others. The conveg-rence at the endpoints $x = \pm 1$ depends on the series.

2. We could also replace x with $-x^2$ in the geometric series.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ converges} \iff -1 < -x^2 < 1 \iff -1 < x < 1$$

   Since $\frac{1}{1+x^2}$ has anti-derivative $\tan^{-1} x$, we have

$$\tan^{-1} x = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

   where $C = 0$ by evaluation at $x = 0$. By the Theorem, this also has radius of convergence 1. In fact, by considering $x = \pm 1$ it is easy to see that the interval of convergence is $[-1, 1]$.

Suggested problems

1. Recall that $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ when $|x| < 1$.

   (a) Find a power series representation for $f(x) = \dfrac{1}{1+x^4}$.

   (b) Use your answer to (a) to find an infinite series expression for the integral

   $$\int_0^{1/3} \frac{1}{1+x^4}\,\mathrm{d}x$$

   (c) Why is this method no use for the integral $\int_0^2 \frac{1}{1+x^4}\,\mathrm{d}x$?

2. Starting with the same geometric series in question 1, find a power series representation of the following functions. For which values of x can you be sure that the power series equals the function?

   (a) $\ln(1+x)$

   (b) $(1-x^2)^{-2}$

   (c) $x\tan^{-1}(3x^2)$

3. (a) Express the power series $\sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n+1}$ in terms of common functions (*Hint: start with* $\frac{1}{1+x^2} \ldots$).

   (b) The function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ was shown in lectures to have radius of convergence ∞. Prove that $f(x) = e^x$ (*Try re-reading section 3.8 from Math 2A…*).