11.9 Representations of Functions as Power Series

As we saw before, one of the main ideas of power series is in trying to represent functions in a different way. The approach really becomes useful when there is no other good way of representing a function. Everything in this section follows from the important fact about geometric series:

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if and only if} \quad -1 < x < 1
\]

**Example** Given the geometric series formula above, we can replace \(x\) with something more complicated, as long as we think about the domain carefully. For instance

\[
\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n} = 1 - x^3 + x^6 - x^9 + \ldots
\]

This equality holds if and only if \(-1 < -x^3 < 1 \iff -1 < x < 1\).

The real benefit is that we can integrate and differentiate this expression, provided \(x\) stays between \(\pm 1\). Therefore

\[
\int \frac{1}{1+x^3} \, dx = \sum_{n=0}^{\infty} (-1)^n \int x^{3n} \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} x^{3n+1}
\]

\[
= C + x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \ldots
\]

again, provided that \(-1 < x < 1\). This integral could have been computed using partial fractions methods, but it is very tricky and slow: a power series representation is much easier. The following theorem formalizes this term-by-term integration and differentiation.

**Theorem.** If \(p(x) = \sum_{n=0}^{\infty} c_n (x-a)^n\) has radius of convergence \(R\), then

\[
p'(x) = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1} = \sum_{n=0}^{\infty} c_{n+1} (n+1)(x-a)^n
\]

\[
\int p(x) \, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1} = C + \sum_{n=1}^{\infty} \frac{c_{n-1}}{n} (x-a)^n
\]

Moreover, both have radius of convergence \(R\).

The integral and derivative don’t necessarily have the same interval of convergence, just radius\(^1\)

**Example** Find the interval of convergence of \(p(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n\), and its integral and derivative.

Applying the ratio test for power series, we have

\[
R = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^2 3^{n+1}}{n^2 3^n} \right| = 3
\]

\(p(x), \int p(x) \, dx\) and \(p'(x)\) therefore all converge absolutely when \(|x| < 3\) and diverge when \(|x| > 3\). It remains to check the endpoints of the intervals of convergence.

\(^1\)The best we can say is that if \(p(x)\) converges absolutely at an endpoint of the interval, then so does its integral.
\[ p(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} x^n \]: At \( x = 3 \) we have the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) which converges. At \( x = -3 \) we have \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \) which also converges. Therefore \( p(x) \) has interval of convergence \([-3, 3]\).

\[ \int p(x) \, dx = C + \sum_{n=1}^{\infty} \frac{1}{(n+1)n^2} x^{n+1} \]: At \( x = 3 \) we have the series \( C + \sum_{n=1}^{\infty} \frac{3}{(n+1)n^2} \) which converges by comparison with \( \sum \frac{1}{n^2} \). Similarly \( \int p(x) \, dx \) converges at \( x = -3 \) and therefore has interval of convergence \([-3, 3]\).

\[ p'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} x^{n-1} \]: At \( x = 3 \) we have the series \( \sum_{n=1}^{\infty} \frac{1}{n^3} \) which diverges, while at \( x = -3 \) we have \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \) which converges. Therefore \( p'(x) \) has interval of convergence \([-3, 3]\).

We can also integrate and differentiate well-understood series term by term.

1. The geometric series \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) has radius of convergence \( R = 1 \). Therefore

\[ \ln |1 - x| = \int_0^x \frac{1}{1-t} \, dt = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} x^n \]

Replacing \( x \) with \( x^3 \), it follows that

\[ \ln |1 - x^3| = \sum_{n=1}^{\infty} \frac{1}{n} x^{3n} \]

after which we could integrate again:

\[ \int \ln |1 - x^3| \, dx = C + \sum_{n=1}^{\infty} \frac{1}{n(3n+1)} x^{3n+1} \]

Since the original series has radius of convergence \( R = 1 \), so do all of the others. The convergence at the endpoints \( x = \pm 1 \) depends on the series.

2. We could also replace \( x \) with \( -x^2 \) in the geometric series.

\[ \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \text{ converges } \iff -1 < -x^2 < 1 \iff -1 < x < 1 \]

Since \( \frac{1}{1+x^2} \) has anti-derivative \( \tan^{-1} x \), we have

\[ \tan^{-1} x = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \]

where \( C = 0 \) by evaluation at \( x = 0 \). By the Theorem, this also has radius of convergence 1. In fact, by considering \( x = \pm 1 \) it is easy to see that the interval of convergence is \([-1, 1]\).
Suggested problems

1. Recall that \( \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \) when \(|x| < 1\).

   (a) Find a power series representation for \( f(x) = \frac{1}{1+x^4} \).

   (b) Use your answer to (a) to find an infinite series expression for the integral

   \[ \int_{0}^{1/3} \frac{1}{1+x^4} \, dx \]

   (c) Why is this method no use for the integral \( \int_{0}^{2} \frac{1}{1+x} \, dx \)?

2. Starting with the same geometric series in question 1, find a power series representation of the following functions. For which values of \( x \) can you be sure that the power series equals the function?

   (a) \( \ln(1+x) \)

   (b) \( (1-x^2)^{-2} \)

   (c) \( x \tan^{-1}(3x^2) \)

3. (a) Express the power series \( \sum_{n=0}^{\infty} (-1)^n (n+1)x^{2n+1} \) in terms of common functions (Hint: start with \( \frac{1}{1+x^2} \ldots \)).

   (b) The function \( f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) was shown in lectures to have radius of convergence \( \infty \). Prove that \( f(x) = e^x \) (Try re-reading section 3.8 from Math 2A \ldots ).