5 Integration

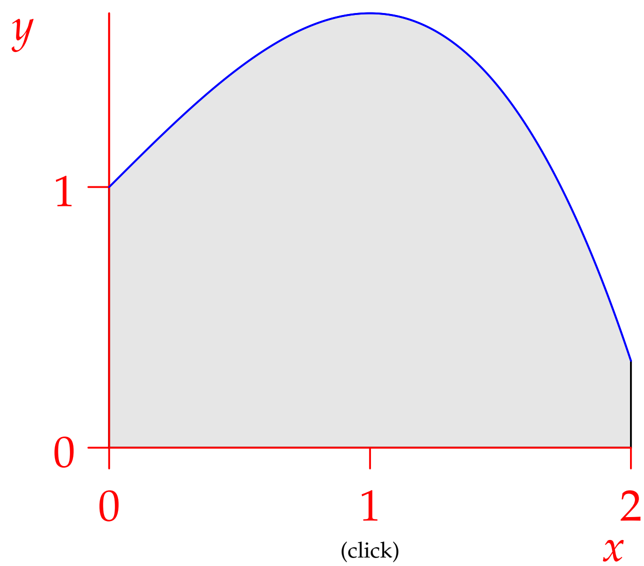
5.1 Areas and Distances

The starting point of integral calculus is the problem of calculating *area.* The naïve concept of area comes from the formula for the area of *rectangle*:

$$\text{Area} = \text{Length} \cdot \text{Width}$$

The area of a triangle is then immediately half that of a rectangle, and any shape which may be subdivided into triangles may have its area computed.

The primary question of integral calculus is of how to extend this idea to cover shapes which cannot be built from triangles. For example, below is the graph of the curve $y) = 1 + x - \frac{1}{3}x^3$ between $0 \leq x \leq 2$; how are we to compute the shaded area?
The answer we provide is very old and very simple:[1] approximate the area under the curve by rectangles and sum the areas of these. If we take a larger number of rectangles, we will hopefully obtain a better approximation to the desired area.[2]



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While the process may seem simple, there are many deep questions raised by it, for instance:

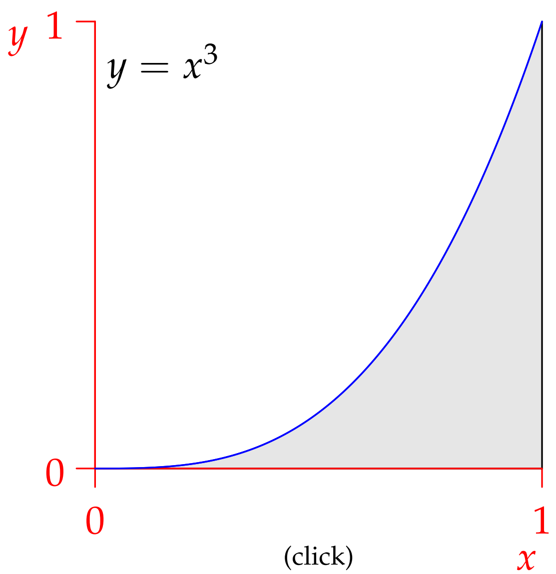
- Does this approximating process work for all functions?
- Does it matter *how* we choose the rectangles?
- The process is inefficient. While a computer might be able to evaluate the sum of the areas of (say) 1500 rectangles in a few seconds, no human can do so. The process will, at best, return only an approximation of the area. Can the area ever be computed *exactly*, or is this a method that should be left to computers?

We cannot properly address all these questions in this course though we will provide partial answers to all three. In particular, let us start with an easier example...

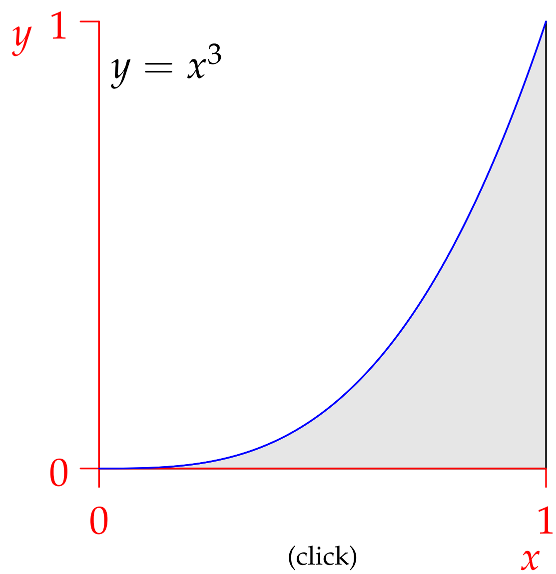
[1] Versions of this method were known to the ancient Greeks 2300 years ago.
[2] If you are using Acrobat Reader, click the picture and convince yourself.

Upper and lower bounds for areas In the pictures below, the area under the curve $y = x^3$ between $x = 0$ and $x = 1$ is estimated using rectangles. In each picture we do the following:

1. Subdivide the interval $[0, 1]$ into n equal segments, where n is a positive integer.
2. Draw a rectangle above each segment so that the upper edge of the rectangle is either *just above* (first picture) or *just below* (second picture) the curve.
3. Since we know the endpoints of each small segment, the equation $y = x^3$ of the curve gives the height and therefore the area of each rectangle.
4. The computer then sums the areas of the drawn rectangles. In the first case the area under the curve is clearly *less* than the sum R_n of the areas of the rectangles. In the second case the area under the curve is *greater* than the sum S_n of the areas of the rectangles.



n rectangles fitting just over curve
$R_n >$ Area under curve



n rectangles fitting just under curve
$S_n <$ Area under curve

For example, if $n = 2$, then we construct two rectangles, each with width $\frac{1}{2}$. The two rectangles in the first picture will have height $\left(\frac{1}{2}\right)^3$ and 1^3, while in the second picture the rectangles have height 0^3 and $\left(\frac{1}{2}\right)^3$. It follows that

$$R_2 = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^3 + \frac{1}{2} \cdot 1^3 = 0.5625$$
$$S_2 = \frac{1}{2} \cdot 0^3 + \frac{1}{2} \cdot \left(\frac{1}{2}\right)^3 = 0.0625$$

From the animations, it appears as if the sequence (R_n) is *decreasing* towards $\frac{1}{4}$, while the sequence (S_n) is *increasing* towards $\frac{1}{4}$. If this is indeed the case, then an application of the squeeze theorem says that the area under the curve really is $\frac{1}{4}$.

$$\begin{cases} S_n < \text{ Area } < R_n \\ \lim\limits_{n\to\infty} S_n = \frac{1}{4} = \lim\limits_{n\to\infty} R_n \end{cases} \quad \underset{\text{Theorem}}{\overset{\text{Squeeze}}{\Longrightarrow}} \quad \text{Area } = \frac{1}{4}$$

In essence, this is the line of reasoning in the abstract theory of Riemann[3] Integration: in an upper-division Analysis course it will be proved that the above process is completely watertight and will allow one to (abstractly) compute the area between the x-axis and *any* continuous curve defined on a closed bounded interval like $[0, 1]$.

Distances

There are a multitude of applications of the problem of finding areas under curves. The oldest, and perhaps the biggest motivator of calculus is the computation of the distance travelled by an objuect whose speed is known. If an object travels at a constant speed v for a time interval t, then the distance it travels is simply $s = vt$.

Now suppose that we are given a velocity-time graph,[4] and that we estimate the area under this curve using rectangles. The area of each rectangle necessarily has units of

$$\text{(velocity)} \cdot \text{(time)} = \text{(distance)}$$

Indeed the area of each rectangle is precisely the distance a particle woule have travelled if its speed had been constant over the indicated time interval. We illustrate this in the example below. For the present it is sufficient to note that the computation of area under a velocity-time graph is precisely that of the distance travelled by the particle. This is exactly the opposite of the *velocity problem* of differential calculus.

Differential Calculus Velocity = Slope of distance-time graph

Integral Calculus Distance travelled = Area under velocity-time

Example Suppose that a car's speedometer reads the following values

Time t (s)	0	15	30	45	60
Speed v (ft/s)	0	18	25	30	27

(a) Estimate the distance traveled by the car over 60 seconds using four sub-intervals and using the speed at the right endpoint of each subinterval.

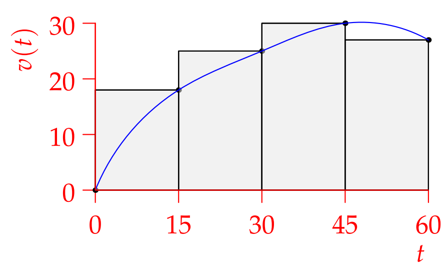
(b) Estimate instead using left endpoints.

(c) Combine (a) + (b) to make a best guess of the distance traveled.

Solution: Each time interval has 'width' 15 seconds. Pretend that car has constant speed over each 15s interval to estimate total distance D traveled.

(a) The speed over each 15s interval is assumed to be the speed at the *end* of each interval. Thus

$$D \approx 15(18 + 25 + 30 + 27) = 1500 \text{ ft}$$

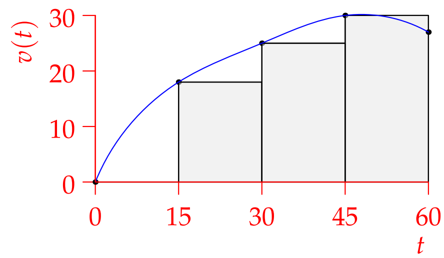


[3]Named after (Georg) Bernhard Riemann (pronounced Reeman!), a German mathematician of the mid 19th century, and the first person to thoroughly justify this process.

[4]Time of the x-axis and velocity on the y-axis.

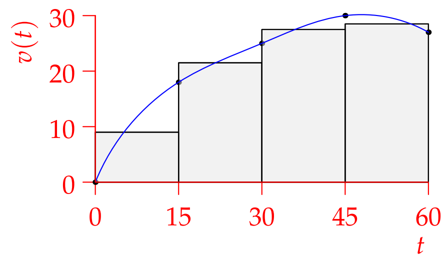
(b) The speed over each 15s interval is assumed to be the speed at the *start* of each interval. Thus

$$D \approx 15(0 + 18 + 25 + 30) = 1095 \text{ ft}$$



(c) For a best guess, we *average* the two results: this is the same as assuming that the speed of the car over each subinterval equals the average of its speed at the start and end of said subinterval.

$$D \approx 15\left(9 + \frac{43}{2} + \frac{55}{2} + \frac{57}{2}\right) = 1297\frac{1}{2} \text{ ft}$$

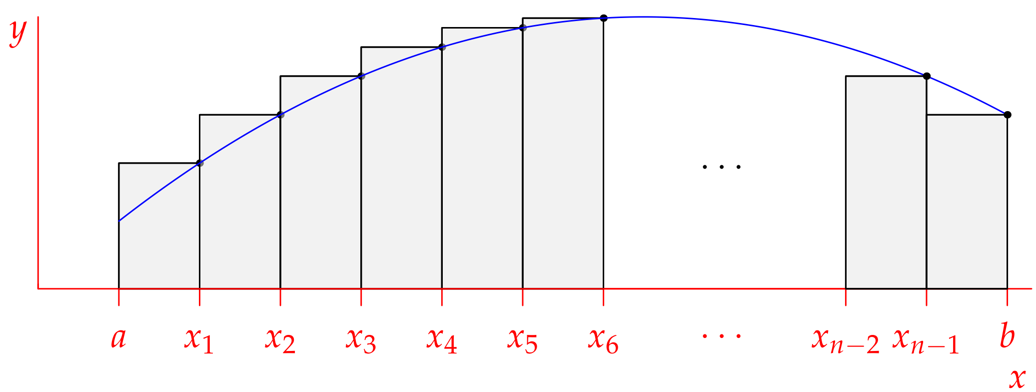


Approximating more generally: Riemann Sums

For any continuous curve $y = f(x)$ between $x = a$ and $x = b$:

1. Subdivide $[a, b]$ into n equal segments of length $\Delta x = \frac{b-a}{n}$
2. Define a sequence $x_i = a + i\Delta x$ $(1 \le i \le n)$ of right endpoints of subintervals
3. Choose a *sample point* $x_i^* \in [x_{i-1}, x_i]$, one in each subinterval
4. The rectangle with base $[x_{i-1}, x_i]$ and height $f(x_i^*)$ has area $f(x_i^*)\Delta x$
5. The area under the curve between $x = a$ and b is then approximately the *Riemann Sum*

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$



Theorem (Very Hard!). *If f is continuous on $[a, b]$, then the limit*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

converges to a value A. This value is independent *of the choice of sample points x_i^*.*

Definition. If f is a non-negative continuous function on $[a,b]$, then the *area* under the curve $y = f(x)$ between $x = a$ and b is defined to be

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = (b-a) \lim_{n\to\infty} \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$$

In particular, because of the Theorem, we can make the simplest choice where each sample point is the right endpoint of each subinterval: $x_i^* = x_i$.

The second displayed expression $\lim_{n\to\infty} \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}$ can be thought of as the *average* height of the function f. We will return to this idea later.

Examples There are two main types of questions involving Riemann Sums. Here we give an example of each.

1. *Evaluate an area using limits.* We compute the area under the curve $y = f(x) = 1 - x^2$ between $x = 0$ and 1 using Riemann Sums.
   Suppose that we have n equal subintervals. Then the width of each is $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. The right-endpoint of each subinterval has x-coordinate

   $$x_i = a + i\Delta x = \frac{i}{n}$$

   It follows that the area under the curve is the limit

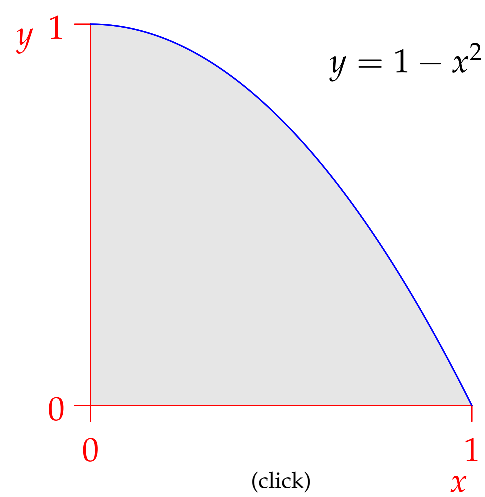
   $$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} (1 - x_i^2)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 - \frac{i^2}{n^2}\right) \frac{1}{n}$$

   Often this is where such a question finishes, as such limit expressions are rarely computable explicitly. However, this is one of very few examples which may be done, by virtue of the identity[5]

   $$1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$$

   In particular

   $$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 - \frac{i^2}{n^2}\right) \frac{1}{n} \\ &= \lim_{n\to\infty} \left(\frac{1}{n} \sum_{i=1}^{n} 1 - \frac{1}{n^3} \sum_{i=1}^{n} i^2 \right) \\ &= \lim_{n\to\infty} \left(\frac{1}{n} \cdot n - \frac{1}{n^3} \cdot \frac{1}{6} n(n+1)(2n+1) \right) \\ &= \lim_{n\to\infty} \left(1 - \frac{n(n+1)(2n+1)}{6n^3} \right) \\ &= 1 - \frac{2}{6} = \frac{2}{3} \end{aligned}$$



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[5] You do not have to memorize this identity, but you should be able to *use* it.

2. *Identify an area writen as a limit of Riemann Sums.* We identify the area defined by the expression

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left(1 - \frac{i^2}{n^2}\right)^{1/2}$$

One challenge here is that there are *infinitely many* possible shapes whose area could be computed as above. We are looking for a *function* f and an *interval* $[a, b]$ for which

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

In particular, we must have

$$f(x_i)\Delta x = \frac{4}{n}\left(1 - \frac{i^2}{n^2}\right)^{1/2}$$

It is often easiest to simply assume that $a = 0$ and to look for a suitable expression for Δx. Here it seems reasonable to take $\Delta x = \frac{4}{n}$, from which, since $\Delta x = \frac{b-a}{n}$, we obtain $b = 4$. It follows that the right-endpoints of each subinterval are

$$x_i = a + i\Delta x = \frac{4i}{n}, \qquad i = 1, \ldots, n$$

Since we still need

$$f(x_i) = \left(1 - \frac{i^2}{n^2}\right)^{1/2} = \left(1 - \left(\frac{x_i}{4}\right)^2\right)^{1/2}$$

we conclude that $f(x) = \left(1 - \frac{1}{16}x^2\right)^{1/2}$. If we write $y = f(x)$, then we are computing the area under the quarter-ellipse $\frac{x^2}{4^2} + y^2 = 1$ between $x = 0$ and $x = 4$. This is $\frac{1}{4}\pi \cdot 4 \cdot 1 = \pi$.

For an alternative answer, we could have chosen $f(x) = \sqrt{4 - x^2}$, with $[a, b] = [0, 2]$ to obtain the area of a quarter-circle.

Suggested problems

1. Calculate the Riemann sums using both right and left endpoints for the function $f(x) = \frac{1}{x}$ on the interval $[1, 5]$ with $n = 4$ subintervals. Sketch the graph and both Riemann sums. Hence find upper and lower bounds for the area under the curve.

2. Suppose that a vehicle's speed $v(t)$ has the following values

Time t (min)	0	1	2	3	4	5
Speed v (m/s)	2	5	20	15	10	20

   (a) Estimate the distance travelled by the vehicle over the time interval $0 \leq t \leq 6$ using a Riemann sum with *three* subintervals and left endpoints.

   (b) Repeat with *six* subintervals and left endpoints.

   (c) Which estimate do you expect to be more accurate? Why?

3. Determine regions whose areas are equal to the following limits: what are their areas?

   (a) $\lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + \frac{2i}{n}\right] \frac{2}{n}$

   (b) $\lim_{n\to\infty} \sum_{i=1}^{n} \left[1 - \frac{i^2}{n^2}\right]^{1/2} \frac{9}{n}$

4. Compute the area under the curve $f(x) = x + x^2$ for $1 \le x \le 3$ directly from the limit definition. You may not use the Fundamental Theorem of Calculus, but may use the following sums

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1), \qquad \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$$