6.2 Volumes

The idea behind computing volumes is the same as that behind computing area.

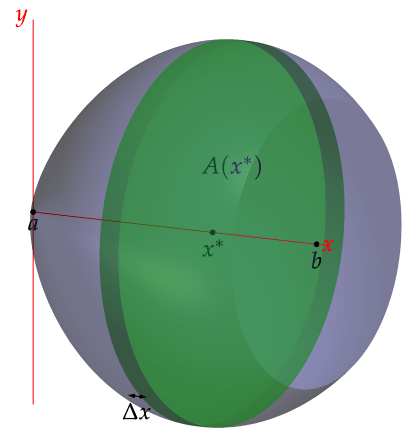
Area Approximate a 2-dimensional region by rectangles and sum their areas.

Volume Approximate a 3-dimensional region by cylinders[1] and sum their volumes.

Suppose that we take the purple solid pictured. We imagine slicing through the solid perpendicular to the x-axis at a distance x^* along said axis. Suppose that we are able to compute the *area* $A(x^*)$ of the resulting cross-sectional slice.
The green cylinder of width Δx has

$$\text{Volume} = A(x^*)\Delta x$$

and approximates part of the original solid. We should therefore be able to approximate the original volume using a Riemann sum of the volumes of many thin cylinders.

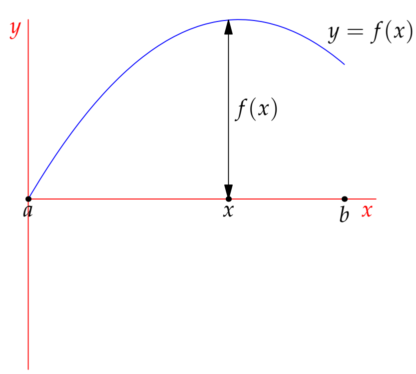


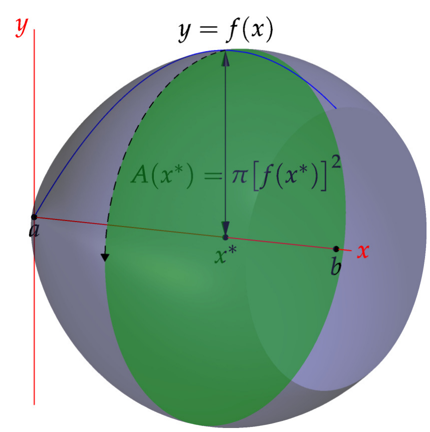
Definition. Suppose that a solid region has cross-sectional area function $A(x)$ whenever $a \le x \le b$. Then its *volume* is the limit of a Riemann sum

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,\mathrm{d}x$$

Volumes of Revolution Most of our examples will be columes obtained by rotating a curve $y = f(x)$ around the x-axis for x in some interval $[a, b]$. It follows that the *radius of rotation* is precisely the value of the function. The pictures should convince you that the cross-sectional area function is

$$A(x) = \pi r^2 = \pi y^2 = \pi\big[f(x)\big]^2$$



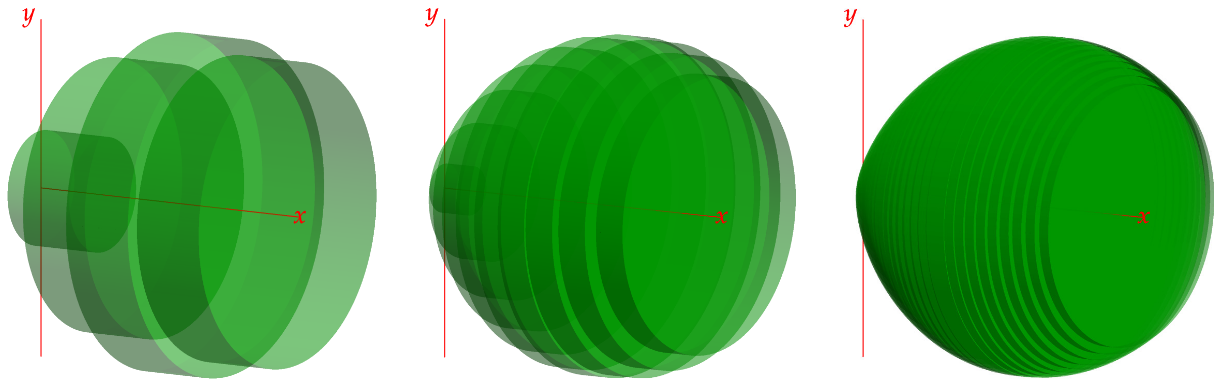


whence the volume is

$$V = \int_a^b A(x)\,\mathrm{d}x = \pi \int_a^b (f(x))^2\,\mathrm{d}x$$

[1] A *cylinder* does not have to be round! It is merely a solid formed by taking a curve and moving it in some direction. Therefore a cube could be described as as square-based cylinder!

The motivating example involved rotating the curve $y = 4x - x^2$ around the x-axis between $x = 0$ and $x = 3$. Below are three approximations with 4, 10 and 30 cylinders.



$V \approx 30.9089\pi$ $\quad$ $V \approx 30.6457\pi$ $\quad$ $V \approx 30.6050\pi$

Since $A(x) = \pi y^2 = \pi(4x - x^2)^2$, the exact volume is

$$V = \int_0^3 A(x)\,\mathrm{d}x = \int_0^3 \pi\left(4x - x^2\right)^2\,\mathrm{d}x = \pi\int_0^3 x^4 - 8x^3 + 16x^2\,\mathrm{d}x$$
$$= \pi\left(\frac{1}{5}\cdot 3^5 - \frac{8}{4}\cdot 3^4 + \frac{16}{3}\cdot 3^3\right) = \frac{153}{5}\pi = 30.6\pi$$

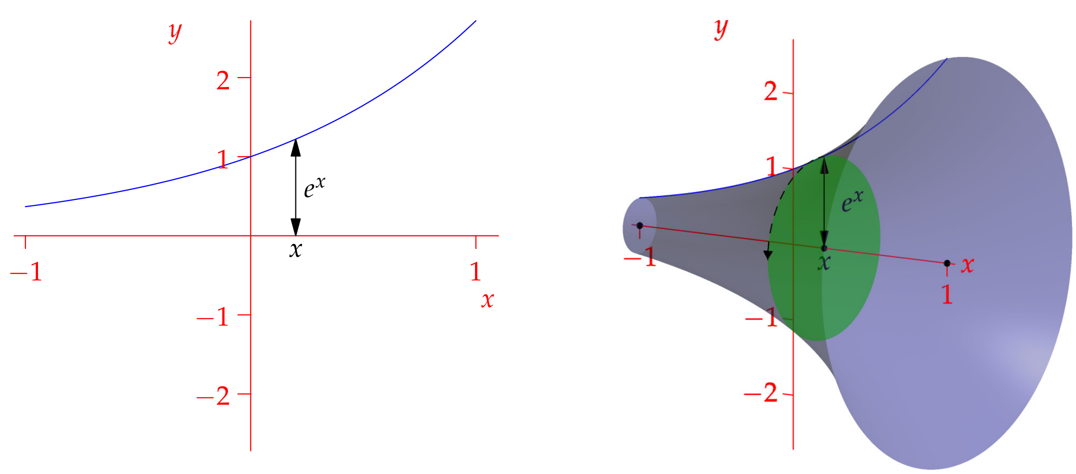
Example Find the volume enclosed when the curve $y = e^x$ is rotated around the x-axis from $x = -1$ to $x = 1$.

The cross-sectional area is

$$A(x) = \pi y^2 = \pi(e^x)^2 = \pi e^{2x}$$

The volume of revolution is therefore

$$V = \pi\int_{-1}^1 e^{2x}\,\mathrm{d}x = \frac{\pi}{2}e^{2x}\Big|_{-1}^1 = \frac{\pi}{2}(e^2 - e^{-2}) \approx 11.394$$

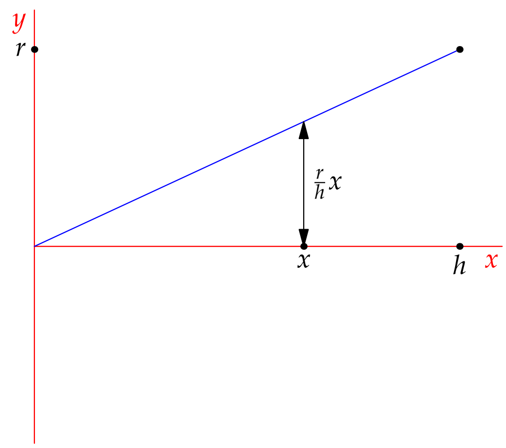


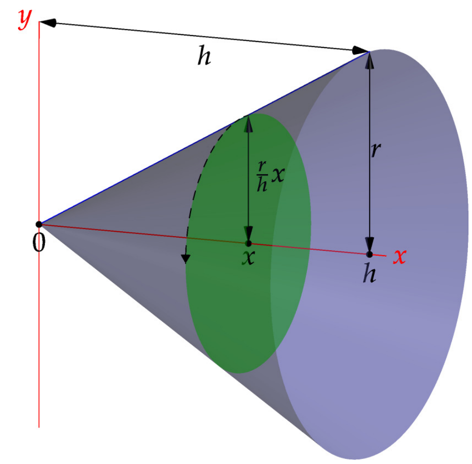
Volume of a right-circular cone We can recover several famous expressions for the volumes of solids using this approach, For example, a cone of height h and base radius r can be formed by rotating the line

$$y = \frac{r}{h}x$$

around the x-axis for $0 \leq x \leq h$. Its volume is therefore

$$V = \pi \int_0^h \frac{r^2}{h^2} x^2 \, \mathrm{d}x = \frac{\pi r^2}{h^2} \cdot \frac{1}{3} x^3 \Big|_{x=0}^{h} = \frac{1}{3}\pi r^2 h$$





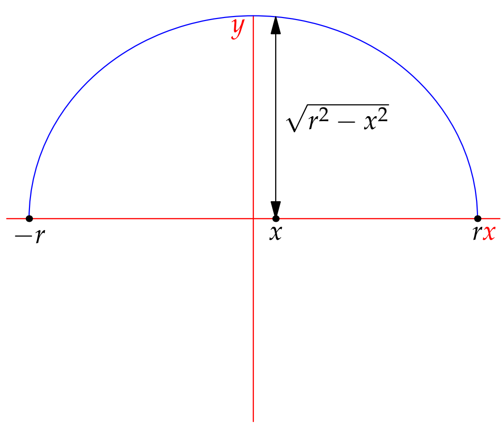
Volume of a sphere A sphere of radius r can be formed by rotating the curve

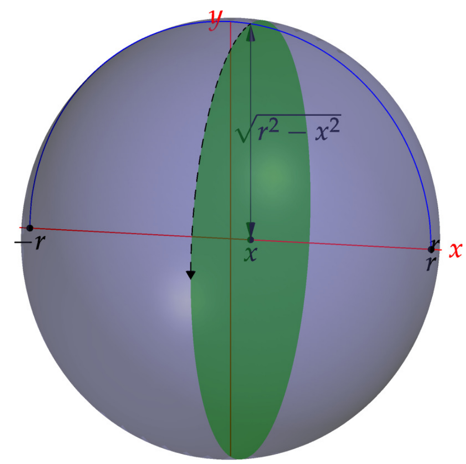
$$y = \sqrt{r^2 - x^2}$$

around the x-axis for $-r \leq x \leq r$. Its volume is therefore

$$V = \pi \int_{-r}^{r} r^2 - x^2 \, \mathrm{d}x = 2\pi \int_0^r r^2 - x^2 \, \mathrm{d}x = 2\pi \left(r^2 x - \frac{1}{3} x^3 \right) \Big|_{x=0}^{r}$$

$$= 2\pi \left(r^3 - \frac{1}{3} r^3 \right) = \frac{4}{3}\pi r^3$$





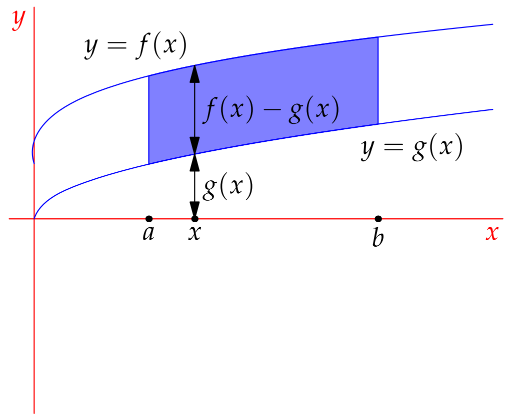
Volumes with annular cross-sections

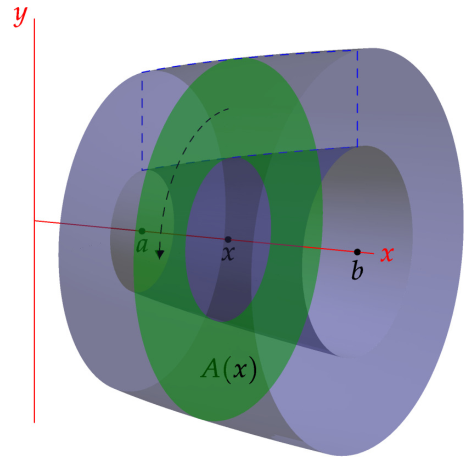
More complicated examples involve rotating the region between two curves $f(x) \geq g(x)$ around the x-axis: the cross-section is an annulus (washer) of area

$$\begin{aligned} A(x) &= \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2 \\ &= \pi(r_{\text{out}}^2 - r_{\text{in}}^2) = \pi(f(x)^2 - g(x)^2) \end{aligned}$$

The volume is therefore

$$V = \pi \int_a^b f(x)^2 - g(x)^2 \,\mathrm{d}x$$

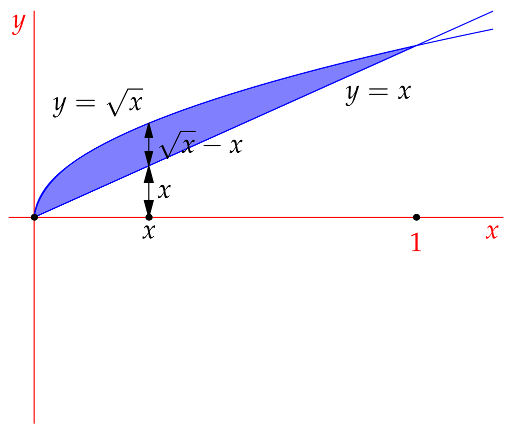


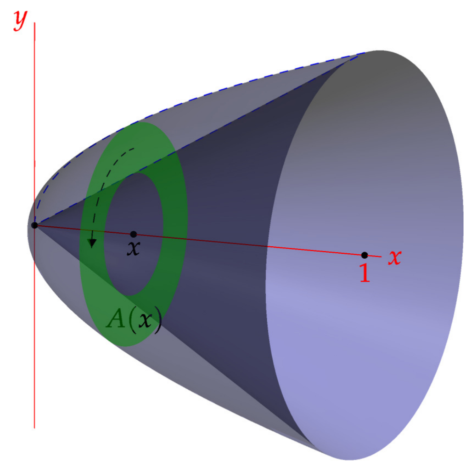


Example Find the volume of the solid given by rotating the region between $y = x$ and $y = \sqrt{x}$ around the x-axis

Since $\sqrt{x} \geq x$ for $0 \leq x \leq 1$ we have

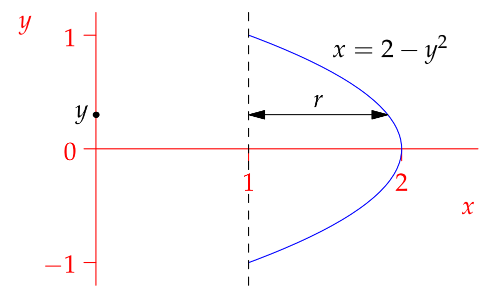
$$V = \pi \int_0^1 (\sqrt{x})^2 - x^2 \,\mathrm{d}x = \pi \int_0^1 x - x^2 \,\mathrm{d}x = \pi \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$$





Revolutions around other axes For even greater complexity, we can revolve curves around the y-axis, or a completely different line: in each case you should find the *radius* (or radii) of revolution and apply one of the methods above.

Example Find the volume of the solid obtained by rotating the curve $x = 2 - y^2$ around the line $x = 1$ between $y = \pm 1$.



The two curves meet when

$$y^2 = 2 - 1 = 1 \implies y = \pm 1$$

The radius of revolution is marked on the picture

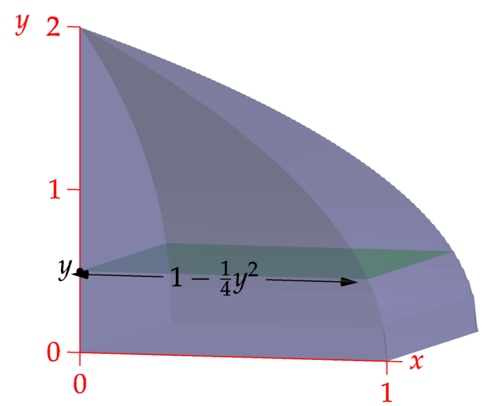
$$r = 2 - y^2 - 1 = 1 - y^2$$

and so

$$V = \int_{-1}^{1} \pi r^2 \, dy = \pi \int_{-1}^{1} (1 - y^2)^2 \, dy = 2\pi \int_0^1 1 - 2y^2 + y^4 \, dy = \frac{16}{15}\pi$$

More general volumes We can use this construction to compute the volume of any object, provided we know its cross-sectional areas. For example, the region below has square horizontal cross-sections of side-length

$$x = 1 - \frac{1}{4}y^2 \quad \text{for} \quad 0 \le y \le 2$$



Its volume is

$$\begin{aligned} V &= \int_0^2 A(y) \, dy = \int_0^2 \left(1 - \frac{1}{4}y^2\right)^2 dy \\ &= \int_0^2 1 - \frac{1}{2}y^2 + \frac{1}{16}y^4 \, dy \\ &= y - \frac{1}{6}y^3 + \frac{1}{80}y^5 \Big|_0^2 = \frac{16}{15} \end{aligned}$$

Suggested problems

1. Let D be the region enclosed by the curves $y = x^2$, $x = 1$, and $y = 0$.
   (a) Rotate D around the x-axis. What is the resulting volume?
   (b) Rotate D around the y-axis. What is the volume now?
2. Rotate the region bounded by the curves $y = 4 - x^2$ and $y = 5 - 2x^2$ around the line $y = 2$. What is the volume?
3. A *torus* (doughnut) is obtained by rotating the region within the circle $(x - R)^2 + y^2 = r^2$ around the y-axis. Assuming $r \le R$, find the volume of the torus.