6.5 The Average Value of a Function

Recall the usual meaning of *average*: if we have a collection of n values $y_1, \ldots, y_n$, then its average is

$$y_{\text{av}} = \frac{y_1 + \cdots + y_n}{n}$$

Now observe that a Riemann sum for a function f on an interval $[a,b]$ is simply the average value of the rectangle-heights $f(x_i^*)$, multiplied by the length of $[a,b]$:

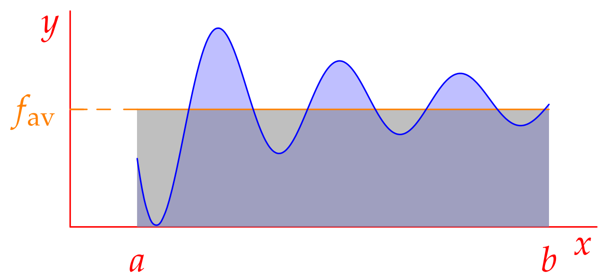
$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n} = (b-a)\frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

This motivates us to *define* the notion of average for any integrable function.

Definition. The average value of a function f over $[a,b]$ is

$$f_{\text{av}} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

To visualize the average value, imaging a bulldozer flattening out all the peaks of a function and pushing the debris into its troughs. The result is a rectangle with height f_{av}. Otherwise said, the blue and gray areas are identical.

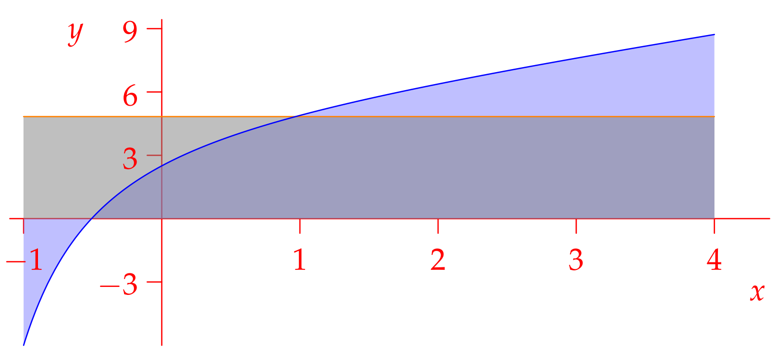


Note that if f is a constant function $f(x) = c$, then the average value is simply

$$f_{\text{av}} = \frac{1}{b-a}\int_a^b c\,\mathrm{d}x = \frac{1}{b-a}(b-a)c = c$$

Example Find the average value of $f(x) = x + 5 - \dfrac{10}{(2+x)^2}$ on the interval $[-1,4]$.

$$f_{\text{av}} = \frac{1}{4-(-1)}\int_{-1}^{4} x + 5 - 10(2+x)^{-2}\,\mathrm{d}x = \frac{1}{5}\left(\frac{1}{2}x^2 + 5x + 10(x+2)^{-1}\right)\Big|_{-1}^{4}$$
$$= \frac{1}{5}\left(\frac{1}{2}(16-1) + 5(4+1) + 10(\tfrac{1}{6} - 1)\right) = \frac{29}{6}$$

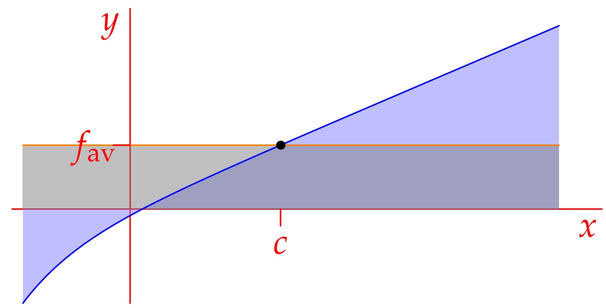


The Mean Value Theorem for Integrals

It seems obvious from the pictures that if a function is continuous on an interval, then its value must at some point equal its average. Consider the following real-world examples.

1. There will always be some time when your velocity equals your velocity speed for a journey.
2. At some point during the day, the temperature will equal the average temperature over that day.

This is indeed a Theorem:



Theorem (Mean Value Theorem for Integrals)**.** *If f is continuous on $[a,b]$, then there exists a number c in (a,b) for which*

$$f(c) = f_{av} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

Proof. Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$, then F is continuous on $[a,b]$ and differentiable on (a,b) by the Fundamental Theorem of Calculus. Applying the Meam Value Theorem to F, we obtain $c \in (a,b)$ for which

$$F'(c) = \frac{F(b)-F(a)}{b-a} = \frac{(b-a)f_{\text{av}} - 0}{b-a} = f_{\text{av}}$$

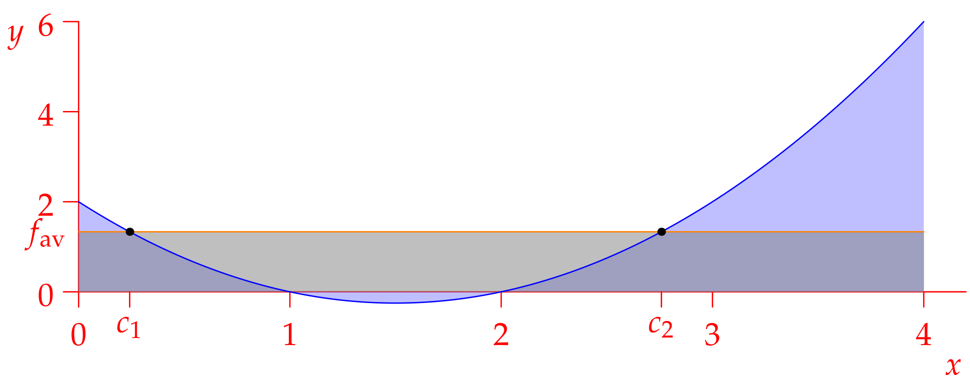
However, by FTC part 1, we have $F'(c) = f(c)$. Hence result. ■

Example If $f(x) = x^2 - 3x + 2$, find all the values c for which $f(c) = f_{\text{av}}$ on $[0,4]$

$$f_{\text{av}} = \frac{1}{4-0}\int_0^4 x^2 - 3x + 2\,\mathrm{d}x = \frac{1}{4}\left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right)\Bigg|_0^4 = \frac{4}{3}$$

We therefore solve

$$f(c) = \frac{4}{3} \iff c^2 - 3c + \frac{2}{3} = 0 \iff c = \frac{3 \pm \sqrt{19/3}}{2} \approx 2.758, 0.242$$



Suggested problems

1. What is the average height of the curve $y = \sqrt{4 - x^2}$, $-2 \leq x \leq 2$, from the x-axis?
2. Let $f(x) = 3x^2 - 2x + 7$ on the interval $[-1, 1]$. Find all values of x for which $f(x) = f_{\text{av}}$.
3. Suppose you walk around a semicircle of radius 2 km at a *constant speed*.[1] What is your average distance from the base of the semicircle?

[1] This is *not* the same problem as question 1...