7.2 Trigonometric Integrals

The three identities $\sin^2 x + \cos^2 x = 1$, $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ and $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ can be used to integrate expressions involving powers of Sine and Cosine. The basic idea is to use an identity to put your integral in a form where one of the substituions $u = \sin x$ or $u = \cos x$ may be applied.

Examples

1. To compute $\int \sin^3 x \, dx$, we first apply the identity $\sin^2 x + \cos^2 x = 1$ to get

$$\int \sin^3 x \, dx = \int \sin x (1 - \cos^2 x) \, dx$$

This is now set up perfectly for the substitution $u = \cos x \implies du = -\sin x \, dx$:

$$\begin{aligned}\int \sin^3 x \, dx &= \int (1 - u^2)(-du) = \int u^2 - 1 \, du \\ &= \frac{1}{3}u^3 - u + c = \frac{1}{3}\cos^3 x - \cos x + c\end{aligned}$$

2. The same trick works in the following:

$$\begin{aligned}\int \sin^3 x \cos^2 x \, dx &= \int \sin x \sin^2 x \cos^2 x \, dx = \int \sin x (1 - \cos^2 x) \cos^2 x \, dx \\ &= \int \sin x (\cos^2 x - \cos^4 x) \, dx \\ &= \int (u^2 - u^4)(-du) \qquad \text{(substitute } u = \cos x) \\ &= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + c = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c\end{aligned}$$

3. With an even power, a double-angle formula is more useful.

$$\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \, dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$$

General Strategies $\int \sin^m x \cos^n x \, dx$

1. If $n = 2k + 1$ is odd, replace $\cos^{2k} x = (1 - \sin^2 x)^k$, then

$$\int \sin^m x \cos^n x \, dx = \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx$$

Now substitute $u = \sin x \implies du = \cos x \, dx$

$$\int \sin^m x \cos^n x \, dx = \int u^m (1 - u^2)^k \, du$$

This is a polynomial in u, which can be integrated after multiplying out.

2. If m is odd, repeat (1) with the roles of cos and sin reversed. Examples 1. and 2. above are of this form.

3. If $m = 2k, n = 2l$ are both even, write $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to obtain

$$\int \sin^m x \cos^n x \, dx = \frac{1}{2^{k+l}} \int (1 - \cos 2x)^k (1 + \cos 2x)^l \, dx$$

which yields integrals of powers of $\cos 2x$. If these powers are odd, then we use the strategies above, if they are even we can repeat to find integrals in terms of $\cos 4x$, etc.
An alternative approach might utilise the identity $\sin x \cos x = \frac{1}{2} \sin 2x$.

Examples

1. To compute $\int \sin^4 x \, dx$ we have to use two identities: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and then $\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$.

$$\begin{aligned}
\int \sin^4 x \, dx &= \int (\sin^2 x)^2 \, dx = \frac{1}{4} \int (1 - \cos 2x)^2 \, dx \\
&= \frac{1}{4} \int 1 - 2\cos 2x + \cos^2 2x \, dx \\
&= \frac{1}{4} x - \frac{1}{4} \sin 2x + \frac{1}{4} \int \cos^2 2x \, dx && \text{(integrate the bits you can...)} \\
&= \frac{1}{4} x - \frac{1}{4} \sin 2x + \frac{1}{8} \int 1 + \cos 4x \, dx && \text{(... use another identity)} \\
&= \frac{3}{8} x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c
\end{aligned}$$

2. If both powers of sine and cosine are odd, you may use either of the first two strategies:

$$\begin{aligned}
\int \sin^5 x \cos^3 x \, dx &= \int \cos x \sin^5 x (1 - \sin^2 x) \, dx = \int \cos x (\sin^5 x - \sin^7 x) \, dx \\
&= \int u^5 - u^7 \, du \quad \text{(substitute } u = \sin x) \\
&= \frac{1}{6} \sin^6 x - \frac{1}{8} \sin^8 x + c_1
\end{aligned}$$

Alternatively

$$\begin{aligned}
\int \sin^5 x \cos^3 x \, dx &= \int \sin x (1 - \cos^2 x)^2 \cos^3 x \, dx = \int \sin x (\cos^3 x - 2\cos^5 x + \cos^7 x) \, dx \\
&= \int u^3 - 2u^5 + u^7 (-\, du) \quad \text{(let } u = \cos x) \\
&= -\frac{1}{4} \cos^4 x + \frac{1}{3} \cos^6 x - \frac{1}{8} \cos^8 x + c_2
\end{aligned}$$

Since both these answers are correct, we must have discovered a (nasty) trig identity! Indeed by evaluating both at $x = 0$ we can quickly see that $c_1 = c_2 - \frac{1}{24}$ and the result is the evil-looking identity

$$4 \sin^6 x - 3 \sin^8 x = 1 - 6 \cos^4 x + 8 \cos^6 x - 3 \cos^8 x$$

3. This time we have a pair of even power. There are again a couple of approaches.

$$\begin{aligned}\int_0^{\pi} \sin^2 x \cos^4 x \, \mathrm{d}x &= \frac{1}{8}\int_0^{\pi} (1-\cos 2x)(1+\cos 2x)^2 \, \mathrm{d}x \\ &= \frac{1}{8}\int_0^{\pi} 1+\cos 2x - \cos^2 2x - \cos^3 2x \, \mathrm{d}x \\ &= \frac{1}{8}\left(\pi - \int_0^{\pi} \frac{1}{2}(1+\cos 4x) + (1-\sin^2 2x)\cos 2x \, \mathrm{d}x\right) \\ &= \frac{1}{8}\left(\frac{\pi}{2} + \int_0^{\pi} \sin^2 2x \cos 2x \, \mathrm{d}x\right) \\ &= \frac{1}{8}\left(\frac{\pi}{2} + \frac{1}{6}\sin^3 2x\Big|_0^{\pi}\right) = \frac{\pi}{16}\end{aligned}$$

or

$$\begin{aligned}\int_0^{\pi} \sin^2 x \cos^4 x \, \mathrm{d}x &= \int_0^{\pi} (\sin x \cos x)^2 \cos^2 x \, \mathrm{d}x = \frac{1}{8}\int_0^{\pi} \sin^2 2x (1+\cos 2x) \, \mathrm{d}x \\ &= \frac{1}{8}\int_0^{\pi} \sin^2 2x \, \mathrm{d}x = \frac{1}{16}\int_0^{\pi} (1-\cos 4x) \, \mathrm{d}x = \frac{\pi}{16}\end{aligned}$$

It is rarely obvious what strategy is going to be best, so don't be surprised if someone finds a faster way to do a problem.

General Strategies for $\int \tan^m x \sec^n x \, \mathrm{d}x$

The next type of trigonometric integrals have the form $\int \tan^m x \sec^n x \, \mathrm{d}x$. These are similar to, but trickier than, the $\int \sin^m x \cos^n x \, \mathrm{d}x$ integrals. The overall goal is exactly the same: use a trig identity to transform the integrand in such a way that an easy substitution will finish things off. The difficulty is that we have fewer easy identities involving tangent and secant to play with.

1. If $n = 2k$ is even take out a factor of $\sec^2 x$. Replace the remaining powers of secant using the identity $\sec^2 x = 1 + \tan^2 !x$, that is

$$\sec^{2k-2} x = (1+\tan^2 x)^{k-1}$$

Our integral is now set up for the substitution $u = \tan x$. The idea is that we have extracted $\sec^2 x = \frac{\mathrm{d}u}{\mathrm{d}x}$ and left everythin else in terms of $u = \tan x$:

$$\begin{aligned}\int \tan^m x \sec^n x \, \mathrm{d}x &= \int \tan^m x (1+\tan^2 x)^{k-1} \sec^2 x \, \mathrm{d}x \\ &= \int u^m (1+u^2)^{k-1} \, \mathrm{d}u\end{aligned}$$

which can be integrated after multiplying out.

2. If $m = 2k+1$ is odd and $n \geq 1$, save a factor of $\sec x \tan x$ and use the identity $\tan^2 x = \sec^2 x - 1$ to convert all the tangents to secants. We can then substitute $u = \sec x$, then $\mathrm{d}u = \sec x \tan x \, \mathrm{d}x$ eliminates the saved expression.

$$\begin{aligned}\int \tan^m x \sec^n x \, \mathrm{d}x &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, \mathrm{d}x \\ &= \int (u^2-1)^k u^{n-1} \, \mathrm{d}u\end{aligned}$$

which can be integrated after multiplying out.

In cases other than the above we may be stuck: we can sometimes appeal to the anti-derivatives

$$\int \tan x \,\mathrm{d}x = \ln|\sec x| + c, \qquad \int \sec x \,\mathrm{d}x = \ln|\sec x + \tan x| + c$$

but these may be insufficient to solve the problem. Integration by parts or some other clever substitution may be necessary.

Examples

1. $$\begin{aligned} \int \tan^2 x \sec^4 x \,\mathrm{d}x &= \int \tan^2 x(1+\tan^2 x)\sec^2 x \,\mathrm{d}x \\ &= \int u^2(1+u^2)\,\mathrm{d}u \quad \text{(substitute } u = \tan x\text{)} \\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + c = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c \end{aligned}$$

2. $$\begin{aligned} \int \tan^5 x \sec^3 x \,\mathrm{d}x &= \int \tan^4 x \sec^2 x \sec x \tan x \,\mathrm{d}x = \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x \,\mathrm{d}x \\ &= \int (u^2-1)^2 u^2 \,\mathrm{d}u \quad \text{(substitute } u = \sec x\text{)} \\ &= \int u^6 - 2u^4 + u^2 \,\mathrm{d}u = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + c \\ &= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + c \end{aligned}$$

3. $$\begin{aligned} \int \tan^2 x \sec x \,\mathrm{d}x &= \int (\sec^2 x - 1)\sec x \,\mathrm{d}x = \int \sec^3 x - \sec x \,\mathrm{d}x \\ &= \int \sec^3 x \,\mathrm{d}x - \ln|\sec x + \tan x| \end{aligned}$$

   $\int \sec^3 x \,\mathrm{d}x$ can be attacked by parts:

   $$\left.\begin{array}{l} u = \sec x \\ \mathrm{d}v = \sec^2 x \,\mathrm{d}x \end{array}\right\} \implies \left\{\begin{array}{l} \mathrm{d}u = \sec x \tan x \,\mathrm{d}x \\ v = \tan x \end{array}\right.$$

   $$\begin{aligned} &\int \tan^2 x \sec x \,\mathrm{d}x = \sec x \tan x - \int \sec x \tan^2 x \,\mathrm{d}x - \ln|\sec x + \tan x| \\ \implies &\int \tan^2 x \sec x \,\mathrm{d}x = \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + c \end{aligned}$$

Other Trigonometric Integrals

- Other trigonometric integrals will not tend to have general strategies: creativity is necessary!
- No other types of trigonometric integral are examinable in this class!

Suggested problems

1. Evaluate the integrals:

   (a) $\int \sin^3 x \cos^2 x \, \mathrm{d}x$

   (b) $\int \cos^4(2x) \, \mathrm{d}x$

2. Evaluate the integrals:

   (a) $\int \sec^4 x \tan^3 x \, \mathrm{d}x$

   (b) $\int_0^{\pi/3} \sec^3 x \tan^3 x \, \mathrm{d}x$

3. The region R_1 is bounded by the graph of $y = \tan x$ and the x-axis on the interval $[0, \pi/3]$. The region R_2 is bounded by the graph of $y = \sec x$ and the x-axis on the interval $[0, \pi/6]$. Which region has the greater area?