7.3 Trigonometric Substitution

These are useful for integrating square-roots of quadratic expressions. That is, if your integrand contains any terms of the form $\sqrt{ax^2 + bx + c}$, where $a, b, c$ are constant.

We have already seen an example using the substitution $x = \sin \theta$:

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \int \frac{1}{\sqrt{1-\sin^2 \theta}} \cos \theta \, d\theta = \int \frac{\cos \theta}{\cos \theta} \, d\theta = \int d\theta = \theta + c = \sin^{-1} x + c$$

General Strategies

There are three primary types of expression. Each can be simplified by a trigonometric substitution.

1. If the integrand contains $\sqrt{a^2 - x^2}$ let $x = a \sin \theta$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = a \cos \theta \quad \text{and} \quad dx = a \cos \theta \, d\theta$$

   We now have an integral containing sines and cosines, which is (hopefully) amenable to the methods of the last section.

2. If the integrand contains $\sqrt{a^2 + x^2}$ let $x = a \tan \theta$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2 \theta} = a \sec \theta \quad \text{and} \quad dx = a \sec^2 \theta \, d\theta$$

   We now have an integral containing secants and tangents.

3. If the integrand contains $\sqrt{x^2 - a^2}$ let $x = a \sec \theta$ where $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$, then

$$\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2 \theta - a^2} = a \tan \theta \quad \text{and} \quad dx = a \sec \theta \tan \theta \, d\theta$$

   We again have an integral containing secants and tangents.

Examples

1. For our first example of the method, we check an integral that may be easily dispatched via the substitution $u = 4 - x^2$.

$$\int x \sqrt{4 - x^2} \, dx = \int \sqrt{u} (-\frac{1}{2} \, du) = -\frac{1}{3} u^{3/2} + c = -\frac{1}{3} (4 - x^2)^{3/2} + c$$

   Instead we apply the methods of this section: let $x = 2 \sin \theta$, then $dx = 2 \cos \theta \, d\theta$, and

$$\int x \sqrt{4 - x^2} \, dx = \int 2 \sin \theta \sqrt{4 + 4 \sin^2 \theta} \cdot 2 \cos \theta \, d\theta = 8 \int \sin \theta \cdot \cos^2 \theta \, d\theta$$

$$= -\frac{8}{3} \cos^3 \theta + c \quad \text{(substitute $u = \cos \theta$ explicitly if you need to)}$$

$$= -\frac{8}{3} \left( \cos(\sin^{-1} \frac{1}{2}) \right)^3 + c$$

1
This answer is revolting! How can we simplify the \( \cos(\sin^{-1}) \) expression? Since trigonometric functions are defined using right-angled triangles, we draw one with angle \( \theta = \sin^{-1} \frac{x}{2} \). This says that

\[
x = 2 \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}
\]

so we draw our triangle with opposite \( x \) and hypotenuse 2. We want

\[
\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}
\]

which, after applying Pythagoras' Theorem to calculate the length \( \sqrt{4-x^2} \) of the adjacent, gives us our result:

\[
\int x \sqrt{4-x^2} \, dx = -\frac{8}{3} \left( \frac{\sqrt{4-x^2}}{2} \right)^3 + c = -\frac{1}{3} (4-x^2)^{3/2} + c
\]

as before.

2. This time we use \( x = 4 \sin \theta \): then \( dx = 4 \cos \theta \, d\theta \), and

\[
\int \frac{1}{(16-x^2)^{3/2}} \, dx = \int \frac{1}{(16-16 \sin^2 \theta)^{3/2}} \cdot 4 \cos \theta \, d\theta = \int \frac{4 \cos \theta}{16^{3/2} (\cos^2 \theta)^{3/2}} \, d\theta
\]

\[
= \frac{1}{16} \int \sec^2 \theta \, d\theta = \frac{1}{16} \tan \theta + c = \frac{x}{16 \sqrt{16-x^2}} + c
\]

To finish things off we needed another triangle, drawn below.

3. For definite integrals, we can change the limits as we go, so no triangle pictures are necessary. Here we let \( x = \sqrt{2} \sin \theta \) then \( dx = \sqrt{2} \cos \theta \, d\theta \). The limits become \( x = 0 \iff \theta = 0 \) and \( x = \sqrt{2} \iff \theta = \frac{\pi}{4} \), whence

\[
\int_0^{\sqrt{2}} x^3 \sqrt{2-x^2} \, dx = \int_0^{\pi/4} 2 \sqrt{2} \sin^3 \theta \sqrt{2-2 \sin^2 \theta} \cdot \sqrt{2} \cos \theta \, d\theta
\]

\[
= 4 \sqrt{2} \int_0^{\pi/4} \sin^3 \theta \cos^2 \theta \, d\theta = 4 \sqrt{2} \int_0^{\pi/4} (\cos^2 \theta - \cos^4 \theta) \sin \theta \, d\theta
\]

\[
= 4 \sqrt{2} \left( -\frac{1}{3} \cos^3 \theta + \frac{1}{5} \cos^5 \theta \right) \bigg|_0^{\pi/4} = \frac{8 \sqrt{2}}{15}
\]

Example could alternatively have been done via the substitution \( u = 2-x^2 \): try it!
The Area of an Ellipse

An ellipse with semi-major axis \(a\) and semi-minor axis \(b\) has equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). Its total area is four times the area of the upper-right quadrant: this is the integral

\[
A = 4 \int_0^a b \sqrt{1 - \frac{x^2}{a^2}} \, dx = \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} \, dx
\]

which can be computed using the substitution \(x = a \sin \theta\). Remember to change the limits…

\[
A = \frac{4b}{a} \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta \, d\theta \\
= 4ab \int_0^{\pi/2} \cos^2 \theta \, d\theta \\
= 2ab \int_0^{\pi/2} 1 + \cos 2\theta \, d\theta = \pi ab
\]

Examples with Secant and Tangent Substitutions

1. Let \(x = 3 \sec \theta\) to obtain \(dx = 3 \sec \theta \tan \theta \, d\theta\). Then

\[
\int \frac{\sqrt{x^2 - 9}}{x} \, dx = \int \frac{\sqrt{9 \sec^2 \theta - 9}}{3 \sec \theta} \cdot 3 \sec \theta \tan \theta \, d\theta \\
= 3 \int \tan^2 \theta \, d\theta = 3 \int \sec^2 \theta - 1 \, d\theta \\
= 3 \tan \theta - 3\theta + c \\
= \sqrt{x^2 - 9} - 3 \sec^{-1} \frac{x}{3} + c
\]

The last step requires a triangle.

\[
\frac{x}{3} = \sec \theta = \frac{1}{\cos \theta} = \frac{\text{hypotenuse}}{\text{adjacent}}
\]

so we draw our triangle with hypotenuse \(x\) and adjacent 3. We want

\[
\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{x^2 - 9}}{3}
\]

2. This time we set \(x = 5 \tan \theta\):

\[
\int \frac{1}{x^2 \sqrt{25 + x^2}} \, dx = \int \frac{5 \sec^2 \theta}{25 \tan^2 \theta \cdot 5 \sec \theta} \, d\theta = \frac{1}{25} \int \frac{1}{\tan^2 \theta \cos \theta} \, d\theta = \frac{1}{25} \int \frac{\cos \theta}{\sin^2 \theta} \, d\theta \\
= -\frac{1}{25 \sin \theta} + c = -\frac{\sqrt{25 + x^2}}{25x} + c
\]

Try drawing the required triangle yourself.
More general expressions $\sqrt{Q(x)}$

By completing the square and changing variables, any quadratic $Q(x)$ may be transformed to one of the standard forms.

**Example** By completing the square, $6x - x^2 = 9 - (x - 3)^2$ which, through the substitution $u = x - 3$, yields

\[
\int \frac{x}{\sqrt{6x - x^2}} \, dx = \int \frac{x}{\sqrt{9 - (x - 3)^2}} \, dx = \int \frac{3 + u}{\sqrt{9 - u^2}} \, du = 3 \sin^{-1} \left( \frac{u}{3} \right) - \sqrt{6x - x^2} + c
\]

\[
= 3 \sin^{-1} \left( \frac{x - 3}{3} \right) - \sqrt{6x - x^2} + c
\]

**Suggested problems**

1. (a) Evaluate the integral $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4 - x^2}} \, dx$

   (b) Evaluate $\int \frac{x \, dx}{\sqrt{16 + 4x^2}}$ using a trigonometric substitution. What method would have been easier?

2. Consider the function $f(x) = (9 + x^2)^{-1/2}$ on the interval $[0, 4]$

   (a) Find the area under the curve $y = f(x)$.

   (b) Find the volume when the region under the curve is rotated around the $x$-axis.

   (c) (Hard) Find the volume when the region under the curve is rotated around the $y$-axis.

3. Evaluate the integral $\int \frac{x^2 + 2x + 4}{\sqrt{x^2 - 4x}} \, dx$, $x > 4$ (You may quote the integrals of $\sec \theta$ and $\sec^3 \theta$)