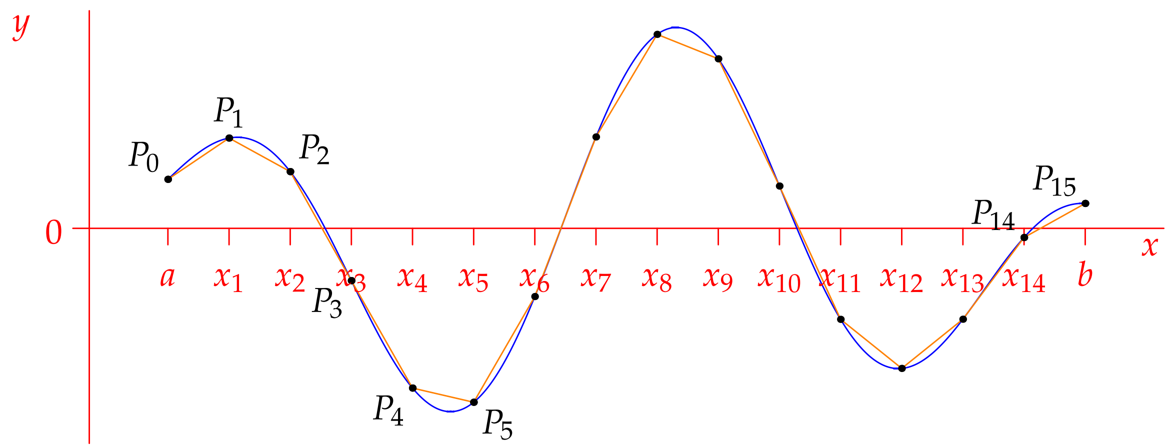
8 Further Applications of Integration

8.1 Arc length

Arc length refers to computing the length of a curve. At its simplest, the idea is nothing more than

$$\text{distance} = \int \text{speed}$$

Suppose you have a particle which travels along a curve $y = f(x)$ between $x = a$ and $x = b$, in such a fashion so that its x-co-ordinate is a measure of *time*. That is, at time t the particle is at the point $(x, y) = (t, f(t))$.



We divide the interval $[a, b]$ into equal subintervals of length $\Delta x = \frac{b-a}{n}$. Viewing x as 'time', we imagine a particle travelling to the right along the curve in such a way that $i\Delta x$ seconds after starting, the particle's location is the point

$$P_i = (x_i, f(x_i))$$

It should be clear that the particle has to move *faster* whenever the curve is *steeper.* Indeed the length of the curve should be approximately the sum of the distances between the points $P_0, \ldots, P_n$:

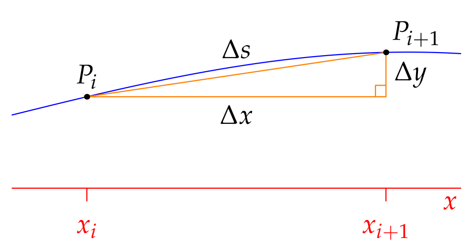
$$\text{Arc-length} \approx \sum_{i=1}^{n} |P_{i-1}P_i|$$

The lengths of these line segments may be computed using Pythagoras' Theorem. Using linear approximations, we see that

$$\Delta y \approx f'(x_i)\Delta x$$

whence

$$\begin{aligned} \Delta s &= \sqrt{(\Delta x)^2 + (\Delta y)^2} && \text{(Pythagoras')} \\ &\approx \sqrt{(\Delta x)^2 + (f'(x_i)\Delta x)^2} && \text{(Linear approximation)} \\ &= \sqrt{1 + [f'(x_i)]^2}\,\Delta x \end{aligned}$$



It follows that the arc-length is approximately the Riemann sum

$$\text{Arc-length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i)]^2}\,\Delta x$$

Since taking more intervals should give a better approximation, we conclude:

Theorem. *If the derivative $f'(x)$ is continuous on the interval $[a, b]$ then the arc-length of the curve $y = f(x)$ between $x = a$ and $x = b$ is the integral*

$$\textit{Arc-length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,\mathrm{d}x$$

One way of interpreting this expression is that if the particle's x-co-ordinate increases by an infinitessimal amount $\mathrm{d}x$, then the distance the particle travels will be *larger* than $\mathrm{d}x$ by the factor $\sqrt{1 + [f'(x)]^2}$ due to the fact that the particle will also move vertically. If we assume that x is measuring time, then the expression $\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + [f'(x)]^2}$ is precisely the speed of the particle so that we really are just integrating speed to get distance.[1]

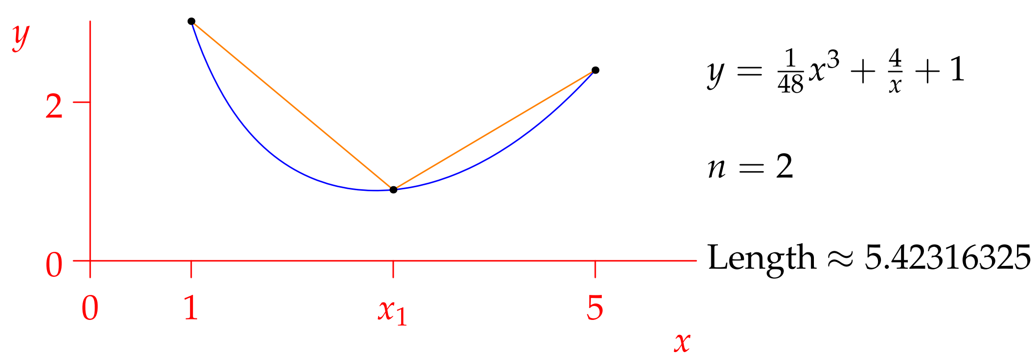
1. The curve $y = f(x) = \frac{1}{48}x^3 + \frac{4}{x} + 1$ is drawn for $1 \le x \le 5$. Since $f'(x) = \frac{1}{16}x^2 - \frac{4}{x^2}$ we have

$$1 + f'(x)^2 = 1 + \frac{1}{256}x^4 - \frac{1}{2} + \frac{16}{x^4} = \left(\frac{1}{16}x^2 + \frac{4}{x^2}\right)^2$$

   It follows that the arc-length of the curve is

$$\int_1^5 \sqrt{1 + f'(x)^2}\,\mathrm{d}x = \int_1^5 \frac{1}{16}x^2 + \frac{4}{x^2}\,\mathrm{d}x = \frac{1}{48}x^3 - \frac{4}{x}\bigg|_1^5 = \frac{31}{12} + \frac{16}{5} = \frac{347}{60} \qquad (\approx 5.7833)$$

   In the animation, the computer is calculating the value of the Riemann sums (the sum of the lengths of the orange lines) for the given number of line segments. Notice how the Riemann sum is always an *underestimate* here.



[1] In multivariable calculus you will consider *parametric curves* where both the x and y-co-ordinates of a point are functions of t. This yields the more symmetric formulæ

$$\text{speed} = \sqrt{x'(t)^2 + y'(t)^2} \implies \text{arc-length} = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t$$

Our formula is simply this one with $x = t$.

2. Similarly $y = f(x) = \frac{1}{32}x^2 - 4\ln x + 7$ for $1 \le x \le 15$ has

$$1 + f'(x)^2 = 1 + \frac{1}{256}x^2 - \frac{1}{2} + \frac{16}{x^2} = \left(\frac{1}{16}x + \frac{4}{x}\right)^2$$

from which the arc-length is

$$\int_1^{15} \sqrt{1 + f'(x)^2}\,\mathrm{d}x = \int_1^{15} \frac{1}{16}x + \frac{4}{x}\,\mathrm{d}x = \frac{1}{32}x^2 + 4\ln x\Big|_1^{15} = 7 + 4\ln 15$$

3. We can use the same approach to confirm the well-known formula for the circumference of a circle. A circle of radius r has equation $x^2 + y^2 = r^2$ from which the circumference is four times the arc-length of the quarter-circle in the positive quadrant: that is

$$\text{Circumference} = 4\int_0^r \sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

where $f(x) = \sqrt{r^2 - x^2}$. Computing this out, we have

$$\begin{aligned}\text{Circumference} &= 4\int_0^r \sqrt{1 + \left[\frac{-x}{\sqrt{r^2 - x^2}}\right]^2}\,\mathrm{d}x = 4\int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\,\mathrm{d}x \\ &= 4\int_0^r \sqrt{\frac{r^2}{r^2 - x^2}}\,\mathrm{d}x = 4r\int_0^r \frac{1}{\sqrt{r^2 - x^2}}\,\mathrm{d}x \\ &= 4r\sin^{-1}\frac{x}{r}\Big|_{x=0}^{r} = 4r\frac{\pi}{2} = 2\pi r\end{aligned}$$

4. A hanging chain is modelled by the curve $y = \frac{e^x + e^{-x}}{2}$ where x lies between $\pm\ln 4$. Its length is then

$$\begin{aligned}\int_{-\ln 4}^{\ln 4} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x &= 2\int_0^{\ln 4} \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}\,\mathrm{d}x \qquad \text{(the integrand is } \textit{even}\text{)} \\ &= 2\int_0^{\ln 4} \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}}\,\mathrm{d}x \\ &= 2\int_0^{\ln 4} \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\,\mathrm{d}x \\ &= 2\int_0^{\ln 4} \frac{e^x + e^{-x}}{2}\,\mathrm{d}x = e^x - e^{-x}\Big|_0^{\ln 4} \\ &= 4 - \frac{1}{4} = \frac{15}{4}\end{aligned}$$

All of these examples are highly contrived so that when we square $f'(x)$ and add 1 we still get a perfect square! If you try to compute the arc-length for a most functions you will be faced with an intergal that is very difficult, if not impossible, to compute exactly. For example, to find the arc-length of the parabola $y = x^2$ from $x = 0$ to 1 requires us to compute the integral

$$\int_0^1 \sqrt{1 + 4x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^{\tan^{-1} 2} \sec^3\theta\,\mathrm{d}\theta$$

where we used the obvious trigonometric substitution $x = \frac{1}{2}\tan\theta$. This integral still requires some tricks! If you look back at the very hard stuff in the trigonometric integrals chapter you might ultimately be able to show that the arc-length is $\frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(2+\sqrt{5})$, but it's very hard work. The moral is that easy functions have difficult arc-length computations: to get a simple arc-length you need to start with a specially designed function!

Suggested problems

1. Evaluate the length of the curve $y = \frac{1}{3}x^{3/2}$ where $0 \leq x \leq 60$.
2. Find the length of the curve $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$ where $1 \leq x \leq 9$.
3. Let $a > 0$ and $A > 0$ be constant. Show that the arc-length integral of the curve

   $$y = Ae^{ax} + \frac{1}{4a^2A}e^{-ax}$$

   for $0 \leq x \leq \ln 2$ is

   $$A(2^a - 1) - \frac{1}{4a^2A}(2^{-a} - 1)$$