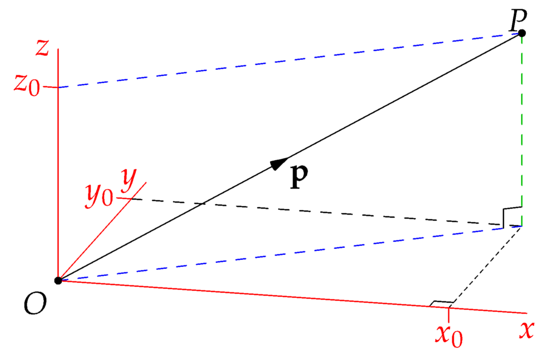
Points and Position Vectors

Point $P = (x_0, y_0, z_0)$ (*co-ordinates* x_0, y_0, z_0)

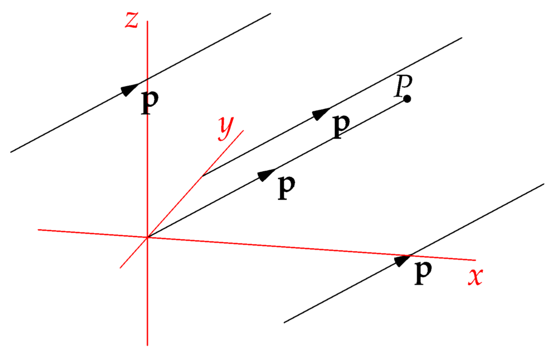
Position vector joins origin O (origin) to P

$$\mathbf{p} = \overrightarrow{OP} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \langle x_0, y_0, z_0 \rangle = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$



$$\textit{Length} = \sqrt{x_0^2 + y_0^2 + z_0^2} = \sqrt{\left(\sqrt{x_0^2 + y_0^2}\right)^2 + z_0{}^2}$$

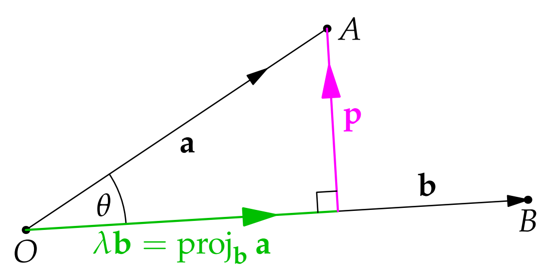
General *vector* **p**: any directed line segment with same length and direction
Does *not* require tail at origin, tip at P



Dot Product and Projection

Given $\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ define

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$



Decompose **a** into component $\lambda\mathbf{b} = \text{proj}_{\mathbf{b}}\, \mathbf{a}$ parallel to **b** and **p** perpendicular. Use Pythagoras to find λ:

$$|\mathbf{a}|^2 = |\lambda\mathbf{b}|^2 + |\mathbf{p}|^2 \iff \lambda = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}$$

Scalar Projection of **a** onto **b**: $\text{comp}_{\mathbf{b}}\, \mathbf{a} = \lambda\, |\mathbf{b}| = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}$

Angle between vectors: $\lambda = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = |\mathbf{a}| \cos\theta \implies \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\, |\mathbf{b}| \cos\theta$

Orthogonality $\theta = 0 \iff \mathbf{a} \cdot \mathbf{b} = 0$

Vector Projection of **a** onto **b**: $\text{proj}_{\mathbf{b}}\, \mathbf{a} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \left(\text{comp}_{\mathbf{b}}\, \mathbf{a}\right) \dfrac{\mathbf{b}}{|\mathbf{b}|}$

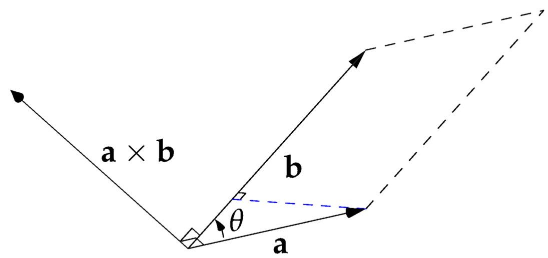
Vector projection has *length* $\text{comp}_{\mathbf{b}}\, \mathbf{a} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = |\mathbf{a}| \cos\theta$, in *direction* of unit vector $\frac{\mathbf{b}}{|\mathbf{b}|}$ parallel to **b**

Work done by force **F** in moving object through displacement **d** is

Cross Product, Right Hand Rule, Area and Volume

Given $\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, define

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

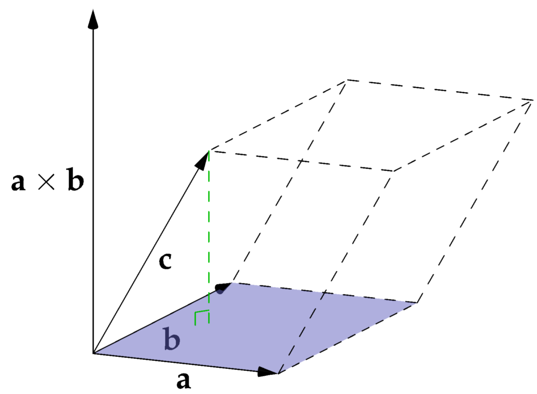


Length $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$ equals[1] area of parallelogram spanned by $\mathbf{a}, \mathbf{b}$

Parallel: $\theta = 0 \iff \mathbf{a} \times \mathbf{b} = \mathbf{0}$

Right-hand Rule: if curl fingers of right hand from $\mathbf{a}$ to $\mathbf{b}$, then thumb points in the direction of $\mathbf{a} \times \mathbf{b}$
`(Some memory aids...)`

Triple Product: $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = |\mathbf{a} \times \mathbf{b}|\, \mathrm{comp}_{\mathbf{a}\times\mathbf{b}}\, \mathbf{c}$
Triple produce equals the (signed) volume of the parallelepiped spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$



Parametrized Lines

To describe a line: choose a *point* P (position vector $\mathbf{r}_0$) and a *direction* vector $\mathbf{v}$
Position vectors of point on the line:

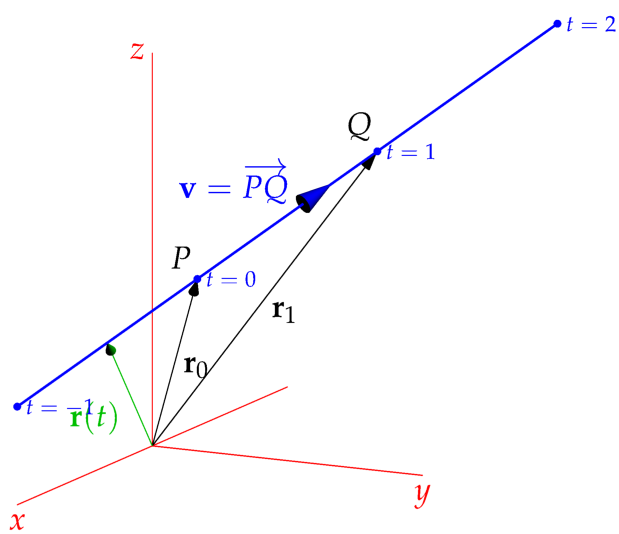
$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} \qquad\qquad (\text{parameter } t \in \mathbb{R})$$

As t changes, tip traces line through P parallel to $\mathbf{v}$

Alternative description: *choose two points P, Q (position vectors $\mathbf{r}_0, \mathbf{r}_1$)*

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$$

To compare: $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0 = \overrightarrow{PQ}$



[1]Multiply out $|\mathbf{a} \times \mathbf{b}|^2$ and compare it to $|\mathbf{a}|^2 |\mathbf{b}|^2 \sin^2\theta = |\mathbf{a}|^2 |\mathbf{b}|^2 (1 - \cos^2\theta) = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$ (horrible!)

Planes

1. Through point P (position vector $\mathbf{r}_0$), perpendicular to $\mathbf{n}$

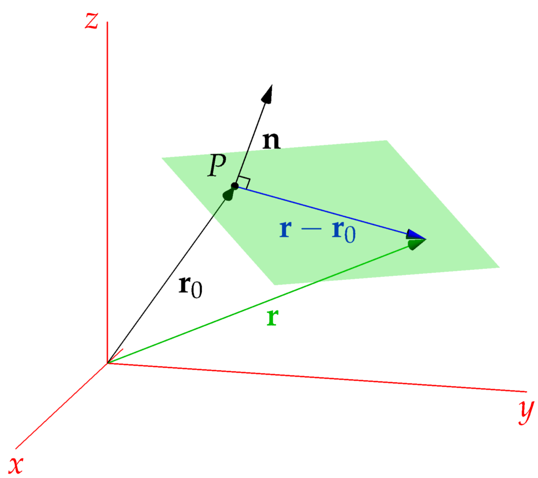
   $$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

   $\mathbf{r} - \mathbf{r}_0$ parallel to plane

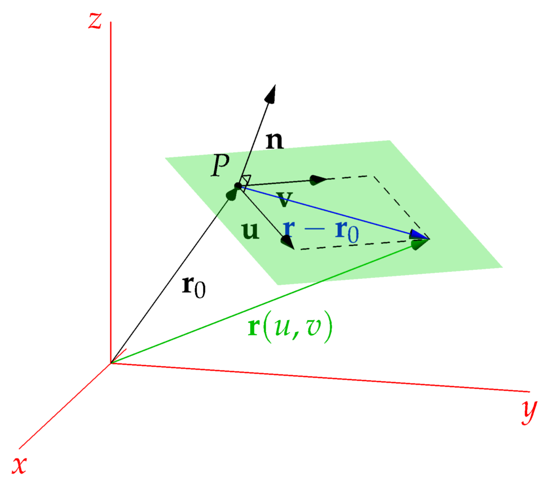
2. (For Math 2E) Through a point P, spanned by constant vectors $\mathbf{u}, \mathbf{v}$

   $$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{u} + v\mathbf{v}$$

   Two parameters $u, v \in \mathbb{R}$



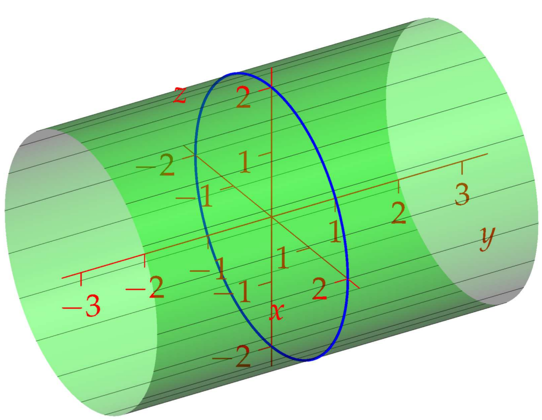
First description



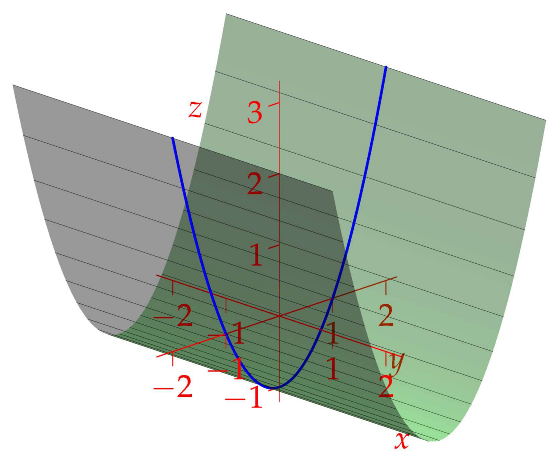
Parametrized (Math 2E) description

Cylinders

Translate a plane curve to form a surface



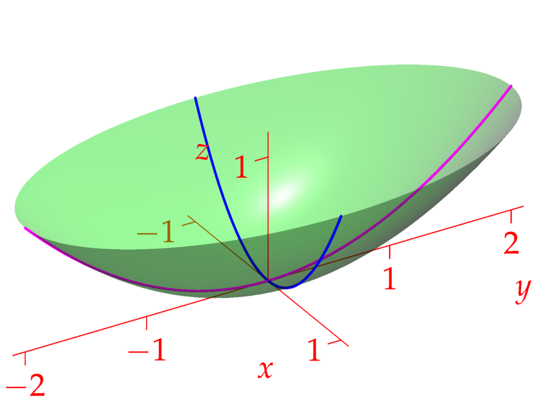
Right-circular cylinder $x^2 + z^2 = 4$



Parabolic cylinder $z = y^2 - 1$

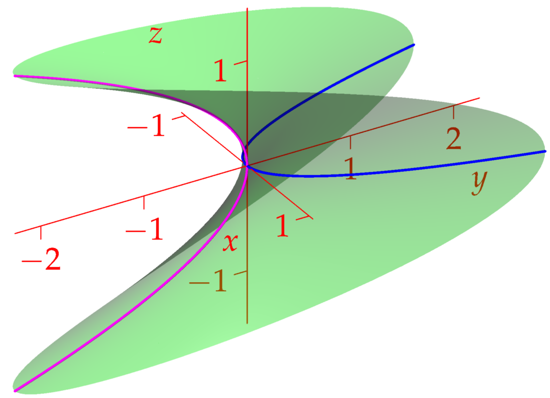
Paraboloids

Use the co-ordinate curves to help visualize/sketch



Elliptic paraboloid $z = x^2 + \frac{y^2}{4}$

$z = x^2$ $\quad z = \frac{y^2}{4}$

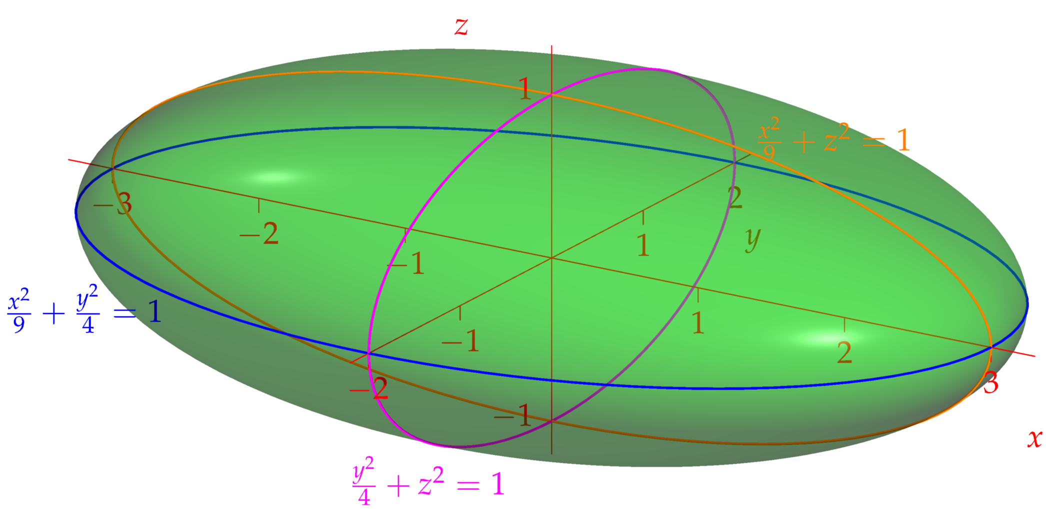


Hyperbolic paraboloid $y = 2x^2 - z^2$

$y = 2x^2$ $\quad y = -z^2$

Ellipsoids

Standard formula: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ describes 'squashed/stretched sphere'

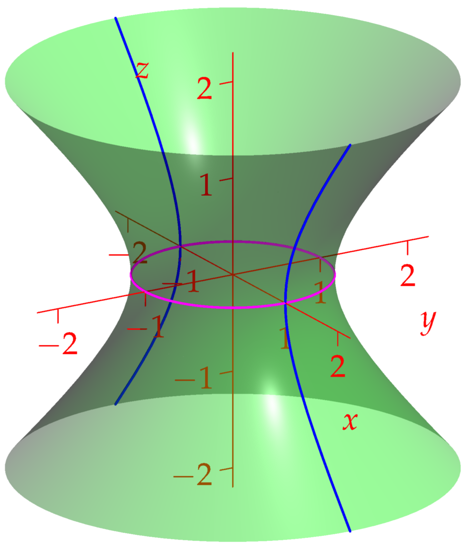


Example $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$: each intersection with a co-ordinate plane is an *ellipse*

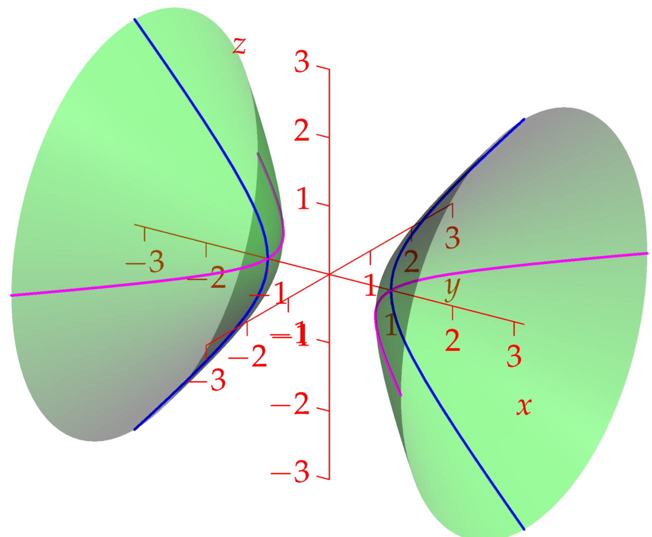
Hyperboloids

Standard formula: $\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$ with one or two negative signs

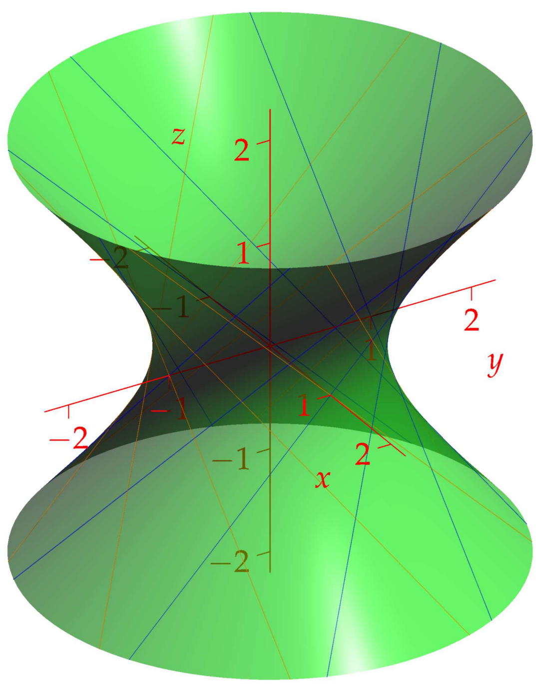
Again use the co-ordinate curves to help visualize/sketch



One sheet (single − sign) $x^2 + y^2 - z^2 = 1$
Circle $x^2 + y^2 = 1$ Hyperbola $x^2 - z^2 = 1$



Two sheets (two − signs) $x^2 - y^2 - z^2 = 1$
Hyperbolæ $x^2 - y^2 = 1$ $x^2 - z^2 = 1$



First example as a doubly-ruled surface