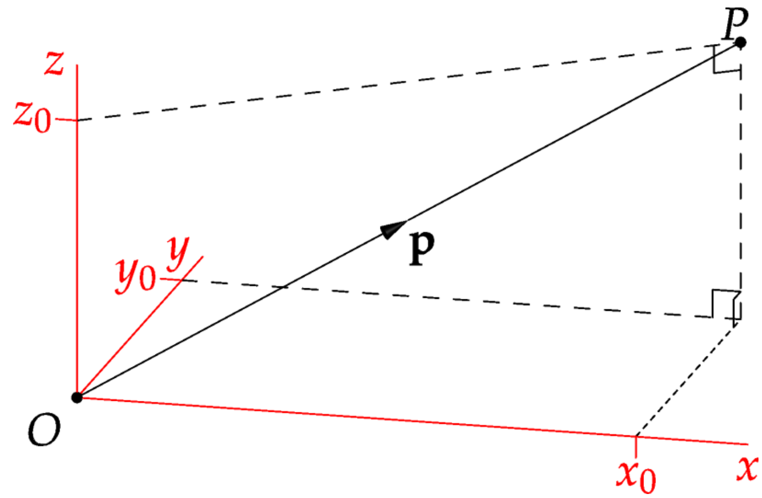
Points and Position Vectors

Point $P = (x_0, y_0, z_0)$ has

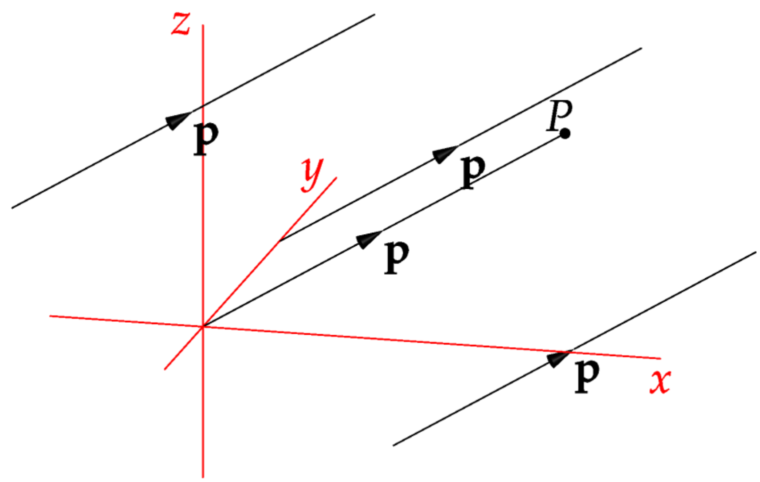
Position vector $\mathbf{p} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$

position vector joins origin O to P



Vector **p** is *any* directed line segment with same length and direction

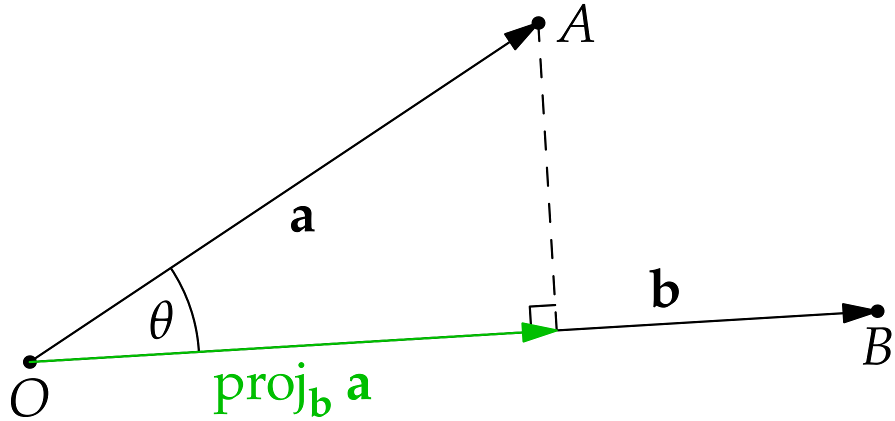
Does *not* require tail at origin, tip at P



Dot Product and Projection

$$\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$$



Projection of $\mathbf{a}$ onto $\mathbf{b}$: $\quad \text{proj}_{\mathbf{b}}\, \mathbf{a} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \left(\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \right) \dfrac{\mathbf{b}}{|\mathbf{b}|}$

$\text{proj}_{\mathbf{b}}\, \mathbf{a}$ has length $\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = |\mathbf{a}| \cos\theta$

$\text{proj}_{\mathbf{b}}\, \mathbf{a}$ points in direction of unit vector $\dfrac{\mathbf{b}}{|\mathbf{b}|}$

Cross Product, Right Hand Rule, Area and Volume

$$\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

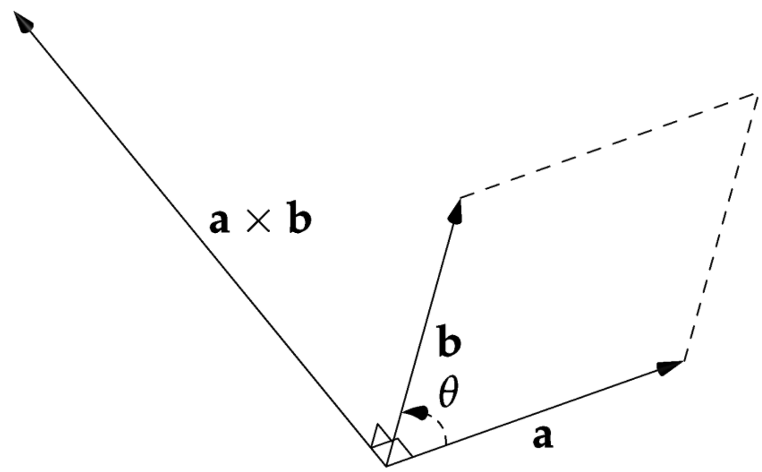
$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Length $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta =$ area of parallelogram spanned by $\mathbf{a}, \mathbf{b}$

Right-hand Rule

Curl fingers of right hand from $\mathbf{a}$ to $\mathbf{b}$: thumb points in direction of $\mathbf{a} \times \mathbf{b}$

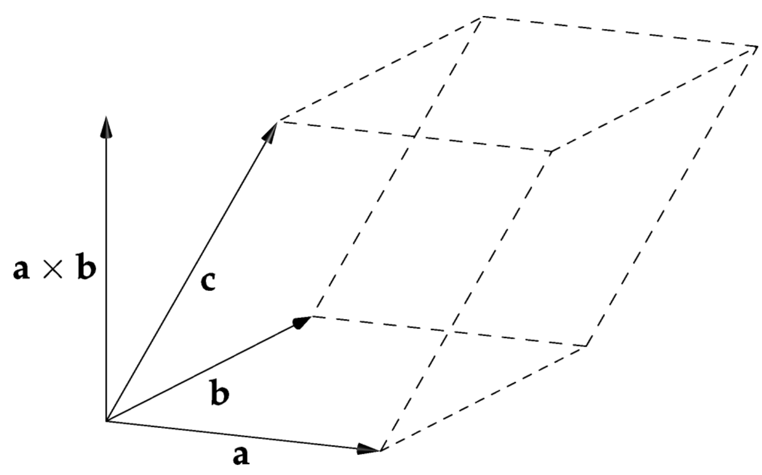
`Some memory aids...`



Parallelepiped spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$

(Signed) Volume $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

Positive if $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ on same side of plane spanned by $\mathbf{a}, \mathbf{b}$



Parameterized Lines

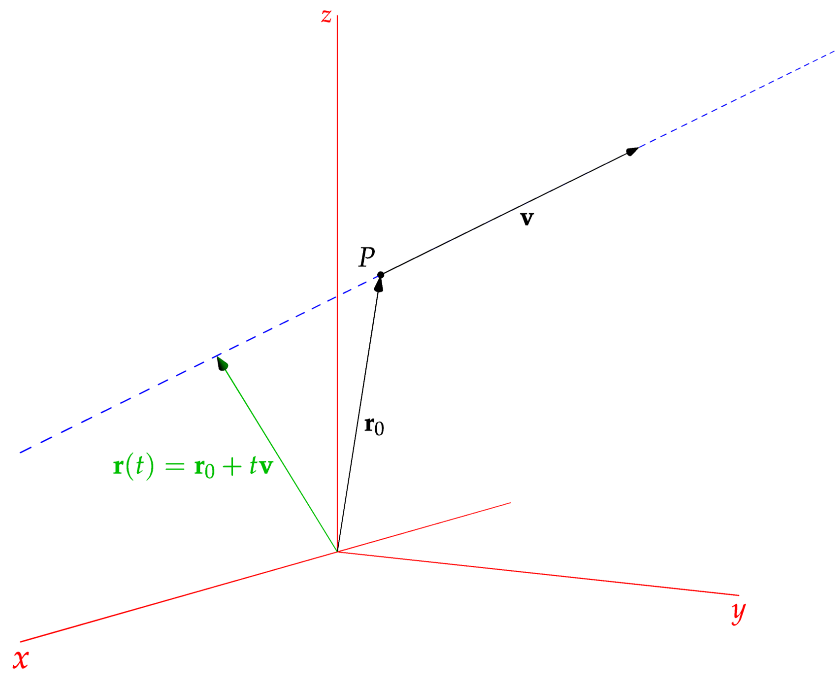
1st description: *through a point P (tip of $\mathbf{r}_0$), parallel to a vector* $\mathbf{v}$

$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$

$\mathbf{r}_0, \mathbf{v}$ *constant* vectors

Parameter $t \in \mathbb{R}$

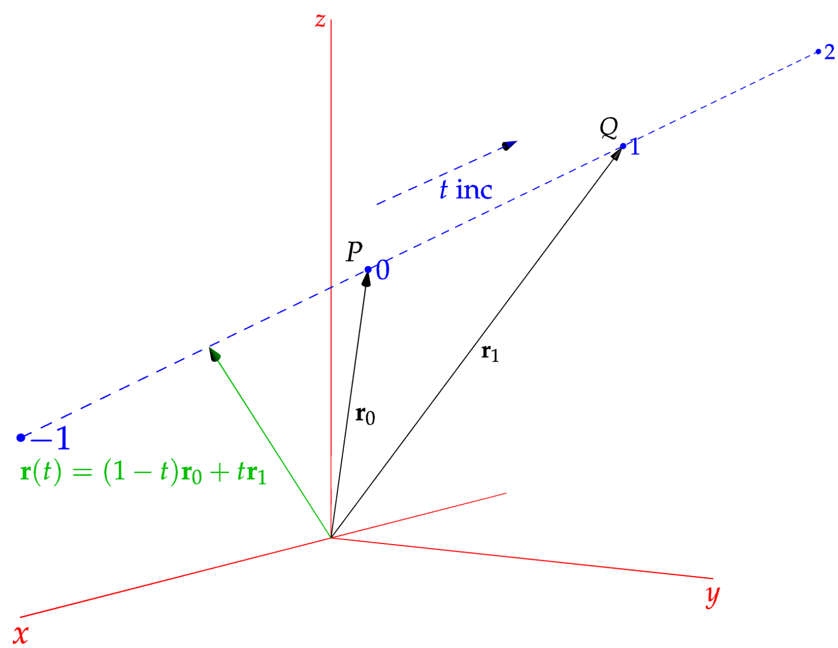
$\mathbf{r}(t)$ moves as t increases: tip traces line parallel to $\mathbf{v}$



2nd description: *through two points P, Q (with position vectors $\mathbf{r}_0, \mathbf{r}_1$)*

$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$

$\mathbf{r}(t)$ moves as t increases: values of t marked in blue



Relationship: $\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0)$, so $\mathbf{v} = \mathbf{r}_1 - \mathbf{r}_0$

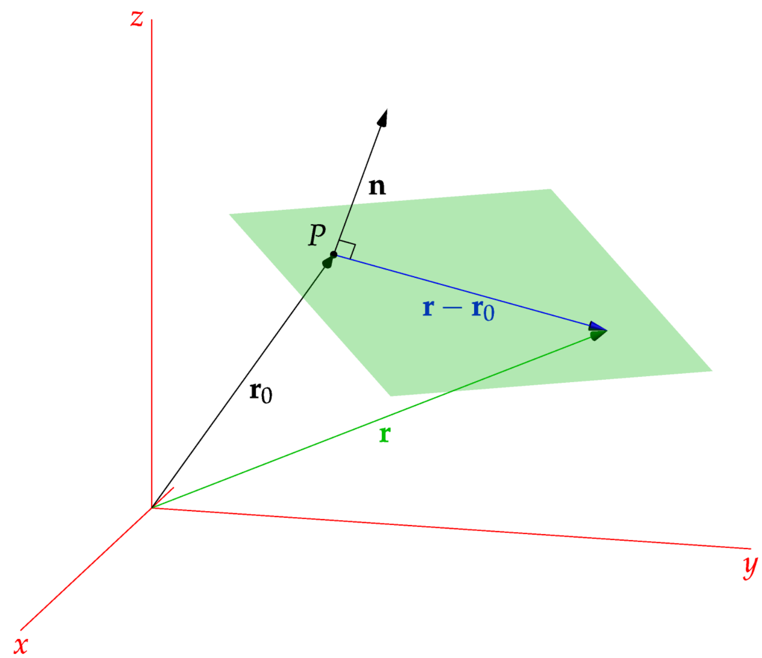
Planes

2D description: *through a point P (tip of* $\mathbf{r}_0$*), perpendicular to vector* $\mathbf{n}$

$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$

$\mathbf{r}_0, \mathbf{n}$ constant vectors

$\mathbf{r} - \mathbf{r}_0$ parallel to plane



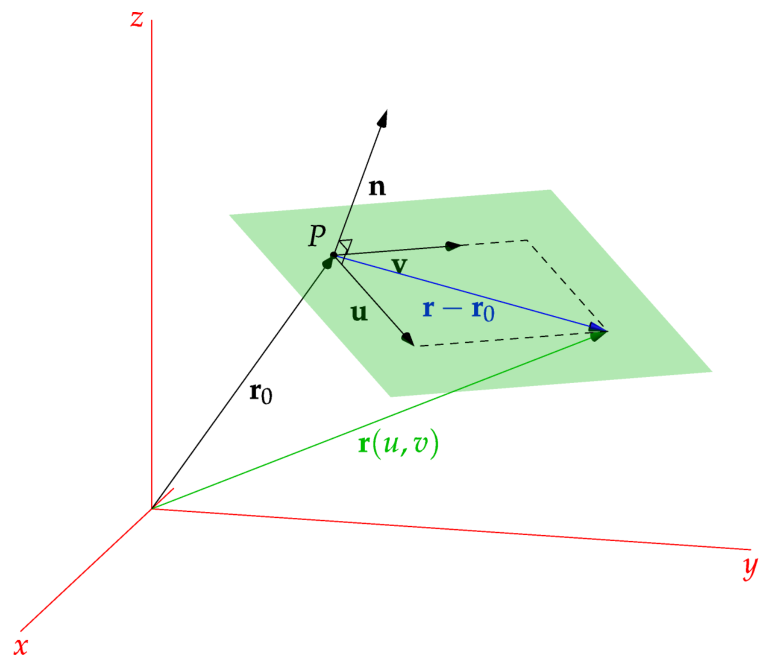
2E description (parameterized): *through a point P, spanned by vectors* $\mathbf{u}, \mathbf{v}$

$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{u} + v\mathbf{v}$

$\mathbf{r}_0, \mathbf{u}, \mathbf{v}$ constant vectors

Two parameters $u, v \in \mathbb{R}$

$\mathbf{r}(u, v) - \mathbf{r}_0$ parallel to plane



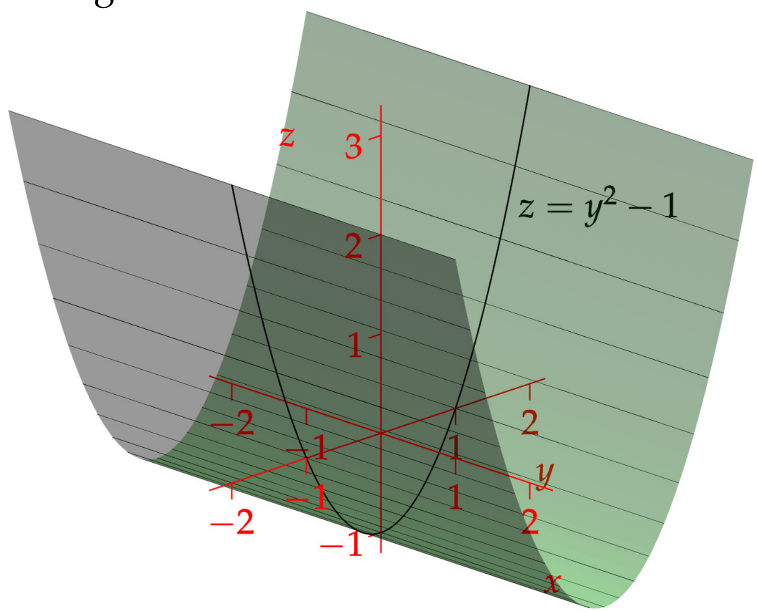
Relationship: $\mathbf{u}, \mathbf{v}$ span plane, $\mathbf{n}$ perpendicular, so $\mathbf{n} \parallel \mathbf{u} \times \mathbf{v}$

Cylinders

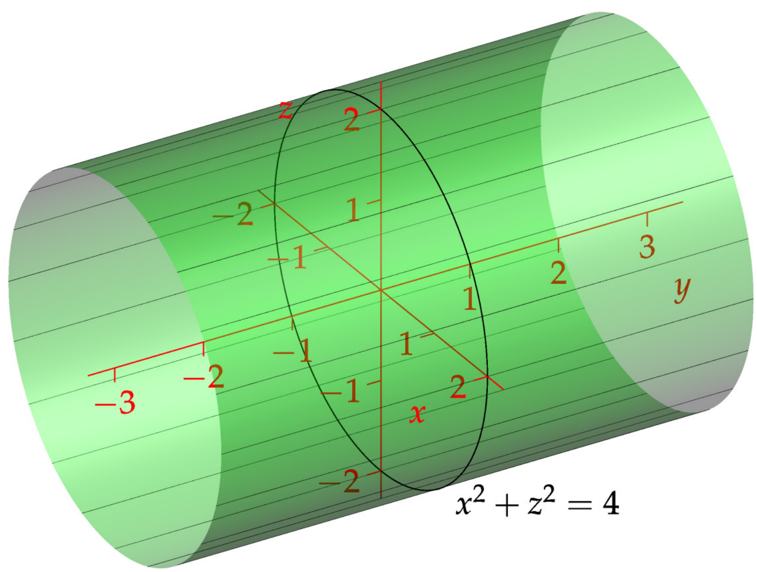
Take a plane curve and translate it perpendicular to the curve

When translating in a co-ordinate direction, the cylinder has *same* equation as the original curve

Parabolic cylinder
$z = y^2 - 1$



Right-circular cylinder
$x^2 + z^2 = 4$



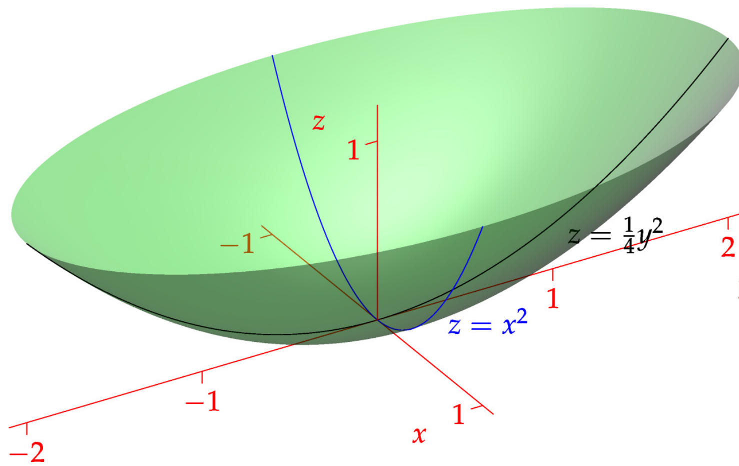
Paraboloids

Typically one variable is a quadratic function of the others

Try graphing the curves when one variable is set to zero

Elliptic paraboloid

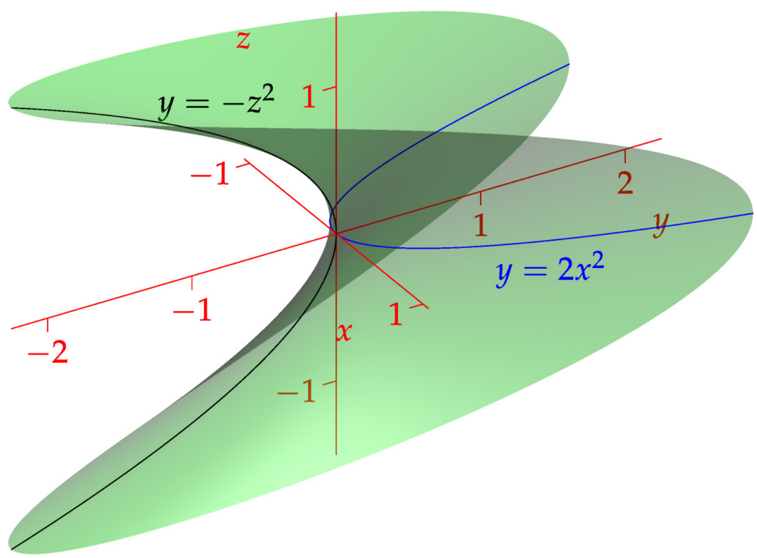
$z = x^2 + \frac{y^2}{4}$



The curves $z = x^2$ and $z = \frac{1}{4}y^2$ help determine the picture

Hyperbolic paraboloid

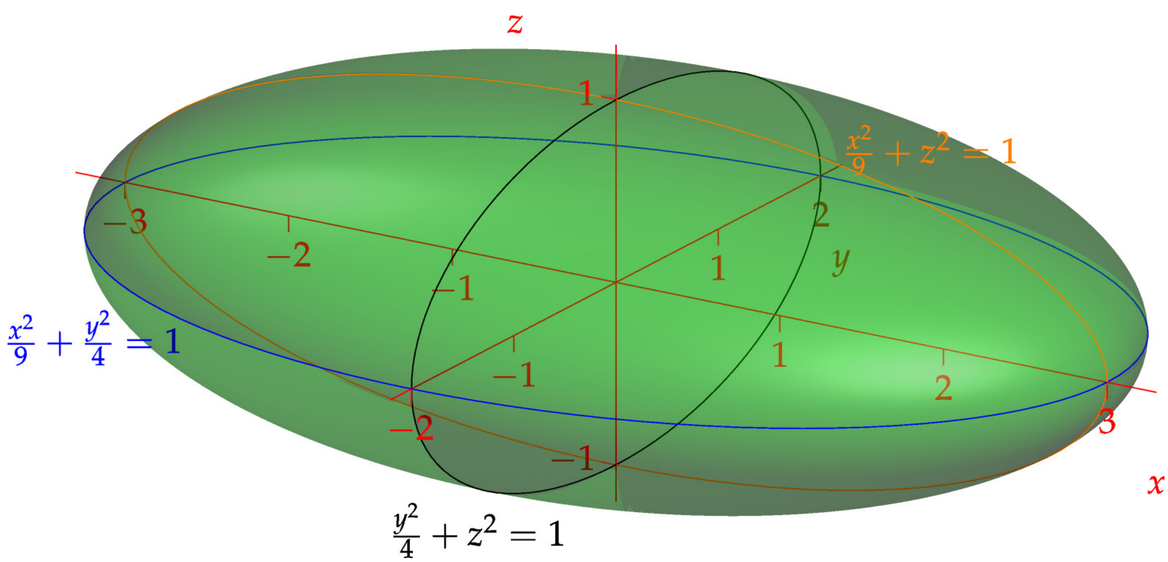
$y = 2x^2 - z^2$



Ellipsoids

Standard formula: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\text{Squashed sphere extending } \begin{cases} -a \le x \le a \\ -b \le y \le b \\ -c \le z \le c \end{cases}$$



Example: $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$

Think about each of the ellipses in the co-ordinate planes

Hyperboloids

Standard formula: $\pm\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$ *with one or two negative signs*

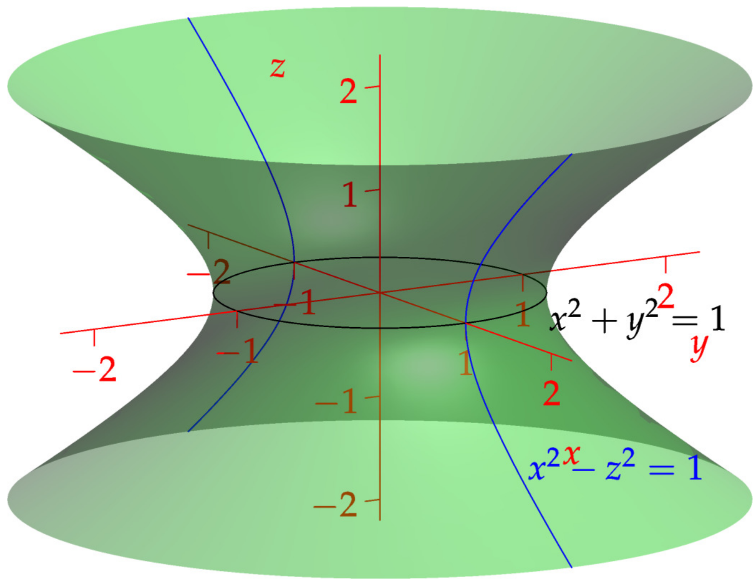
Think about curves in the co-ordinate planes

One sheet (single − sign)

E.g., $x^2 + y^2 - z^2 = 1$

Circle: $x^2 + y^2 = 1$

Hyperbola: $x^2 - z^2 = 1$



Two sheets (two − signs)

E.g., $x^2 - y^2 - z^2 = 1$

Hyperbola: $x^2 - y^2 = 1$

Hyperbola: $x^2 - z^2 = 1$