Continuity in Several Variables

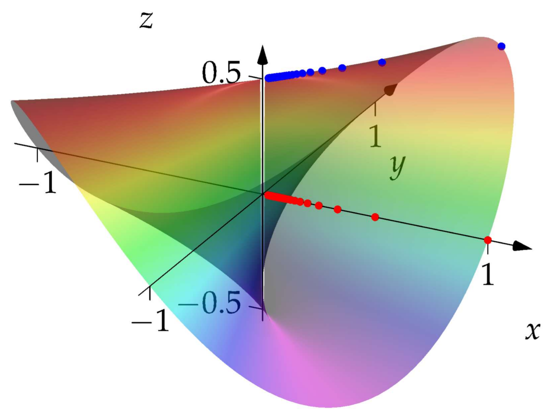
Example 1: $f(x,y) = \frac{xy}{x^2+y^2}$ with two sequences satisfying

$$\lim(x_n, y_n) = (0,0)$$

Red: $(x_n, y_n) = (\frac{1}{n}, 0) \implies f(x_n, y_n) = 0$

Blue: $(x_n, y_n) = (\frac{1}{n}, \frac{1}{n}) \implies f(x_n, y_n) = \frac{1}{2}$

Different limits $\implies \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist and f is *not* continuous at $(0,0)$



Example 2: $g(x,y) = \frac{x^2 y}{x^2+y^2}$ has limit

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0$$

To see this, use polar co-ordinates:

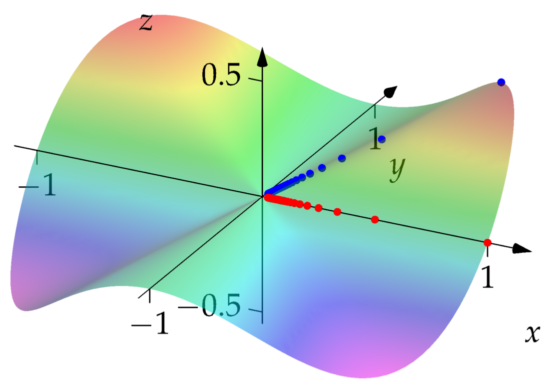
$$g(x,y) = \frac{r^3 \cos^2\theta \sin\theta}{r^2} = r\cos^2\theta\sin\theta$$

$$\implies |g(x,y)| \le r \xrightarrow[r\to 0^+]{} 0$$

All sequences $(x_n, y_n) \to (0,0)$ have the same limit

$$\lim_{n\to\infty} g(x_n, y_n) = 0$$

If we *define* $g(0,0) = 0$, then g is continuous at $(0,0)$.



The Tangent Plane

Example: $z = f(x,y) = -x^2 - \frac{1}{2}y^2 + xy + 2$ at $(x_0, y_0) = (1,2)$

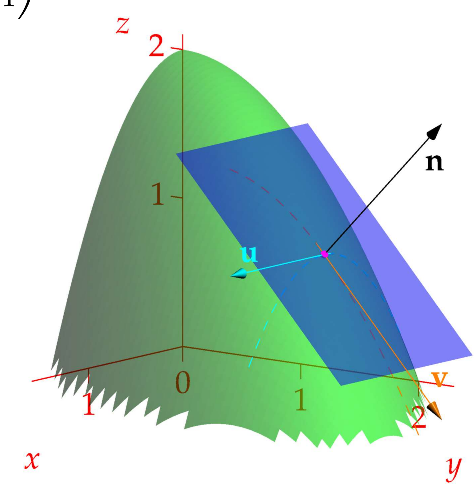
$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ f_x(1,2) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ f_y(1,2) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\mathbf{u}$ is tangent to the curve $\mathbf{r}(t) = \begin{pmatrix} t \\ 2 \\ f(t,2) \end{pmatrix}$ at $t = 1$

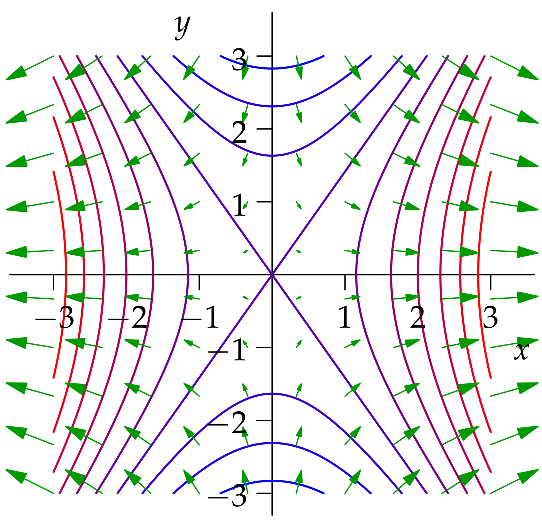
$\mathbf{v}$ is tangent to the curve $\mathbf{r}(s) = \begin{pmatrix} 1 \\ s \\ f(1,s) \end{pmatrix}$ at $s = 2$

If $F(x,y,z) = z - f(x,y)$, then $\mathbf{n} = \nabla F(1,2)$

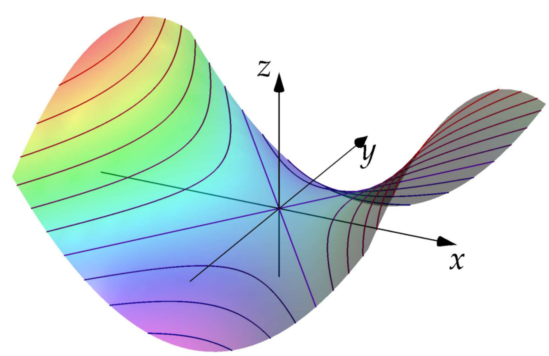


The Gradient Vector

Example: $f(x,y) = \frac{1}{4}x^2 - \frac{1}{8}y^2 \implies \nabla f = \begin{pmatrix} x/2 \\ -y/4 \end{pmatrix}$



Gradient vector field $\perp$ to level curves



Warmer color $\leftrightsquigarrow$ increased height