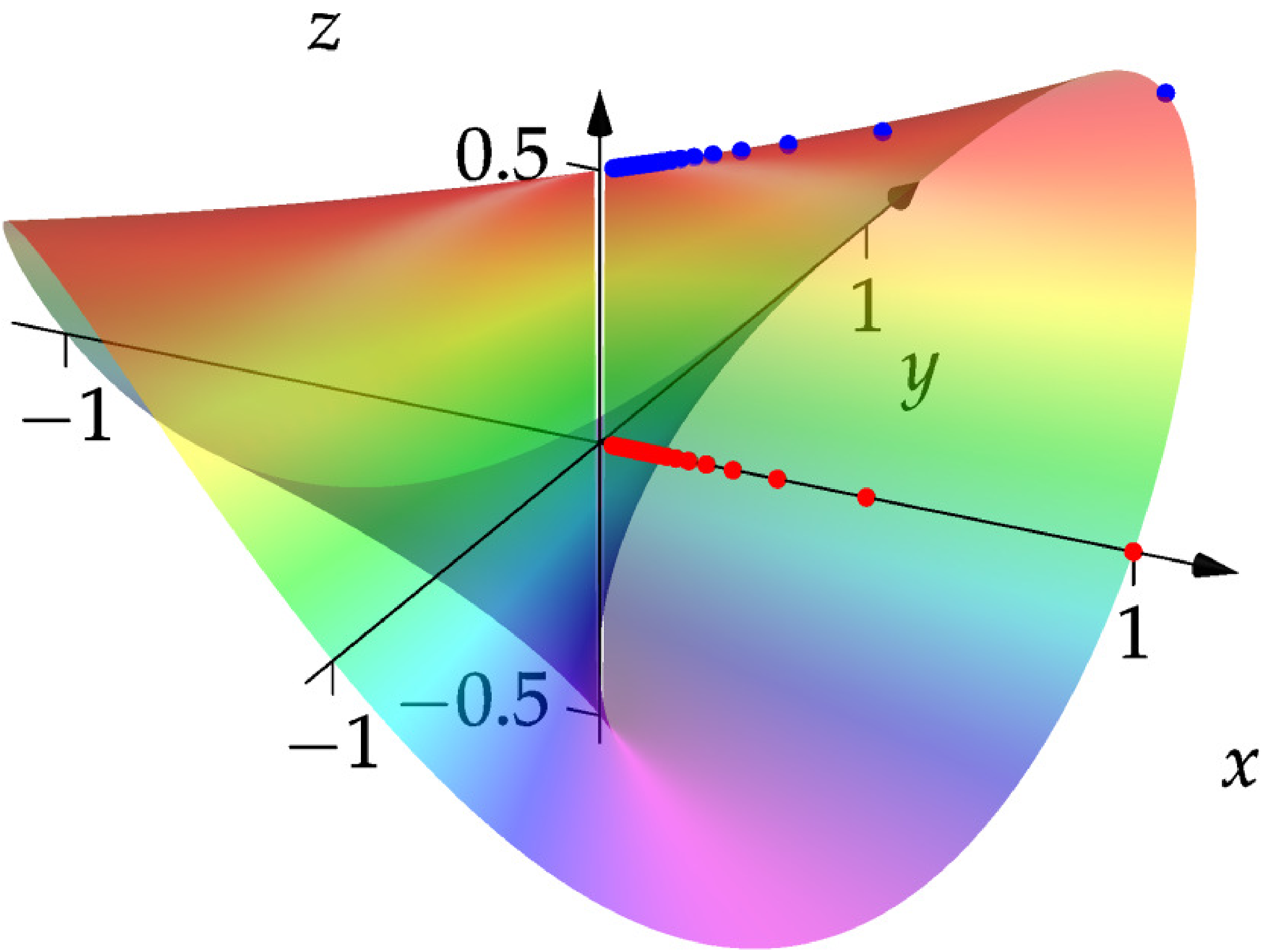
Continuity in Several Variables

$$f(x,y) = \frac{xy}{x^2+y^2}$$



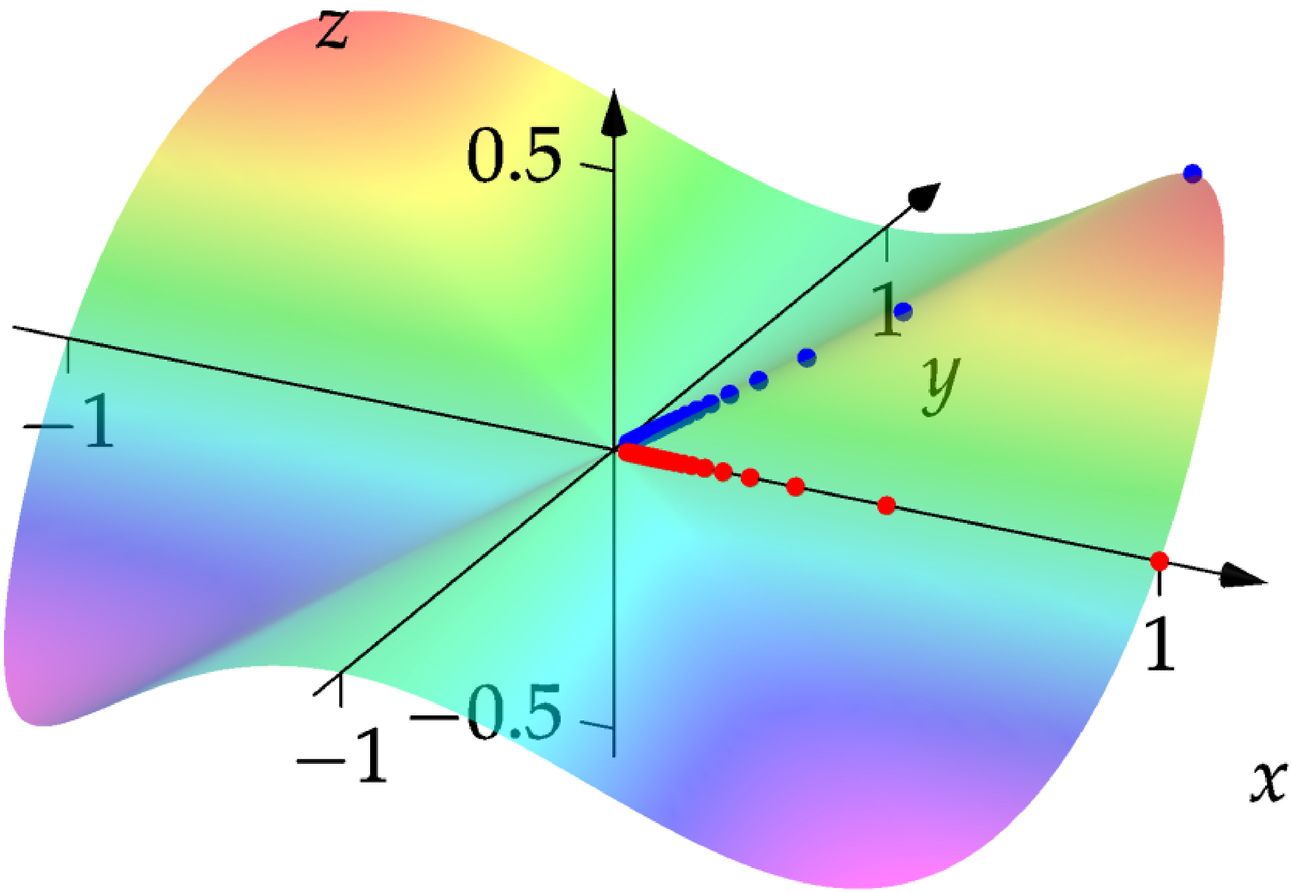
Red sequence: $(x_n, y_n) = (\frac{1}{n}, 0) \implies f(x_n, y_n) = 0$

Blue sequence: $x_n = y_n = \frac{1}{n} \implies f(x_n, y_n) = \frac{1}{2}$

Different limits $\implies \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

f *not* continuous at $(0,0)$

$$f(x,y) = \frac{x^2 y}{x^2 + y^2}$$



All sequences $(x_n, y_n) \to (0,0)$ have same limit for $f(x_n, y_n)$

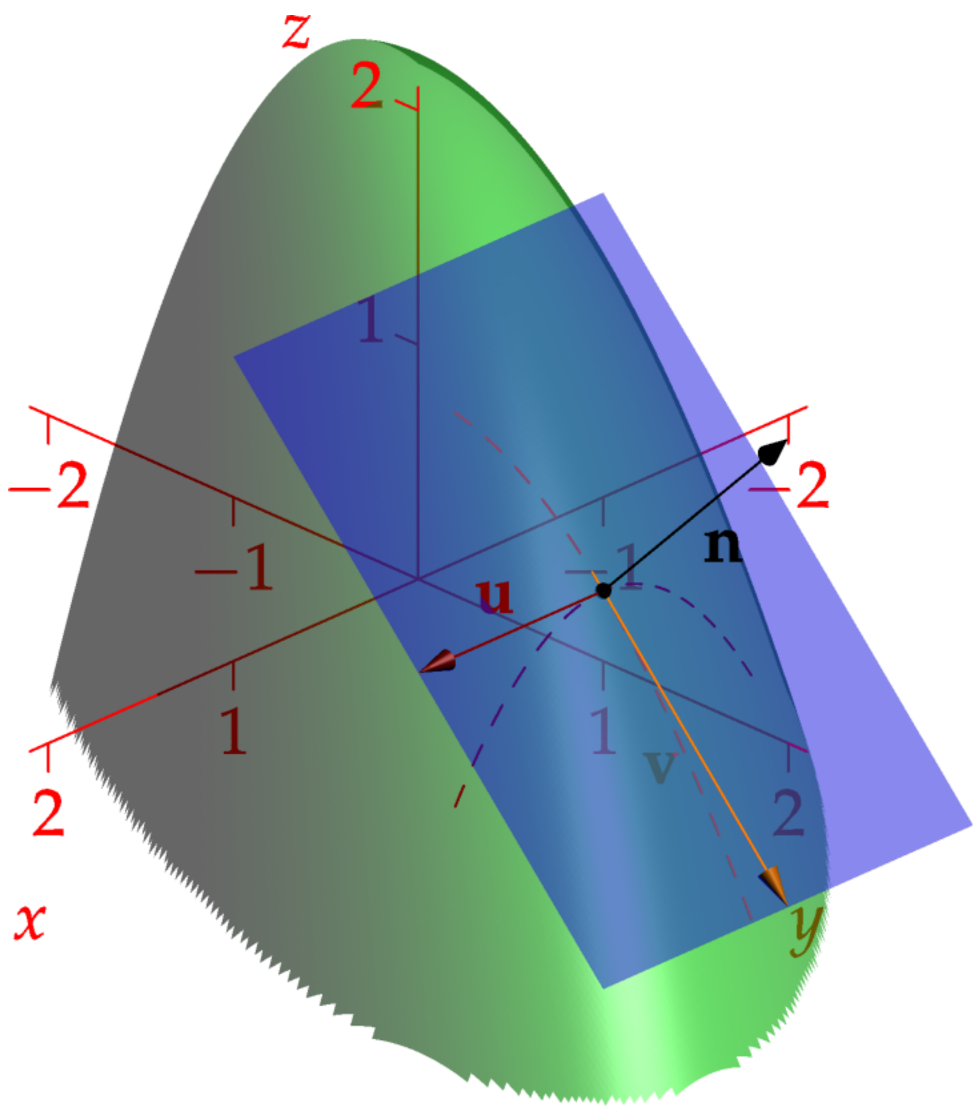
$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

If we *define* $f(0,0) = 0$, then f is continuous at $(0,0)$

The Tangent Plane

$$z = f(x,y) = -x^2 - \tfrac{1}{2}y^2 + xy + 2 \text{ at } (x_0, y_0) = (1,2)$$

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ f_x(1,2) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ f_y(1,2) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$



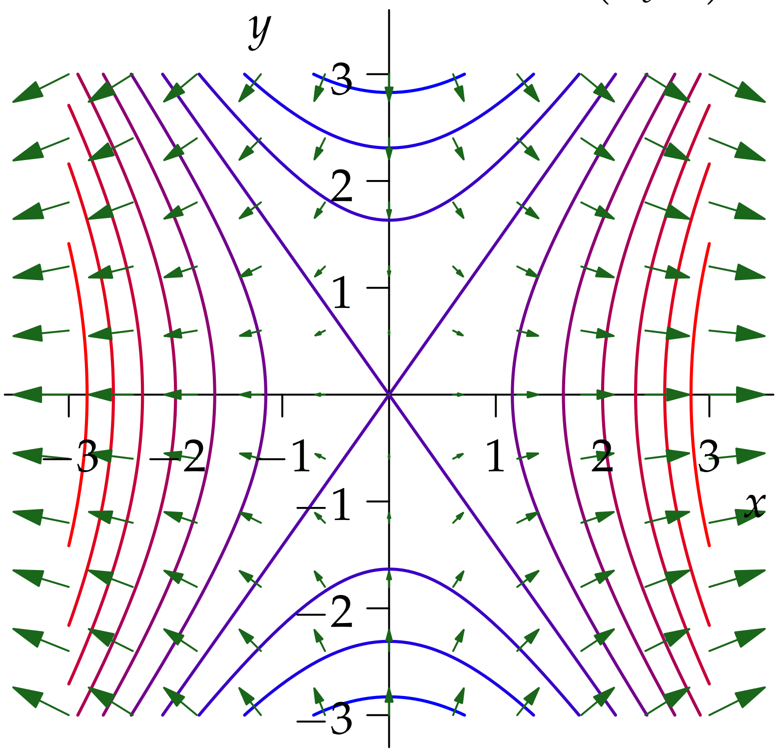
$\mathbf{u}$ is tangent to the curve parameterized by $\mathbf{r}(t) = \begin{pmatrix} t \\ 2 \\ f(t,2) \end{pmatrix}$ at $t = 1$

$\mathbf{v}$ is tangent to the curve parameterized by $\mathbf{r}(s) = \begin{pmatrix} 1 \\ s \\ f(1,s) \end{pmatrix}$ at $s = 2$

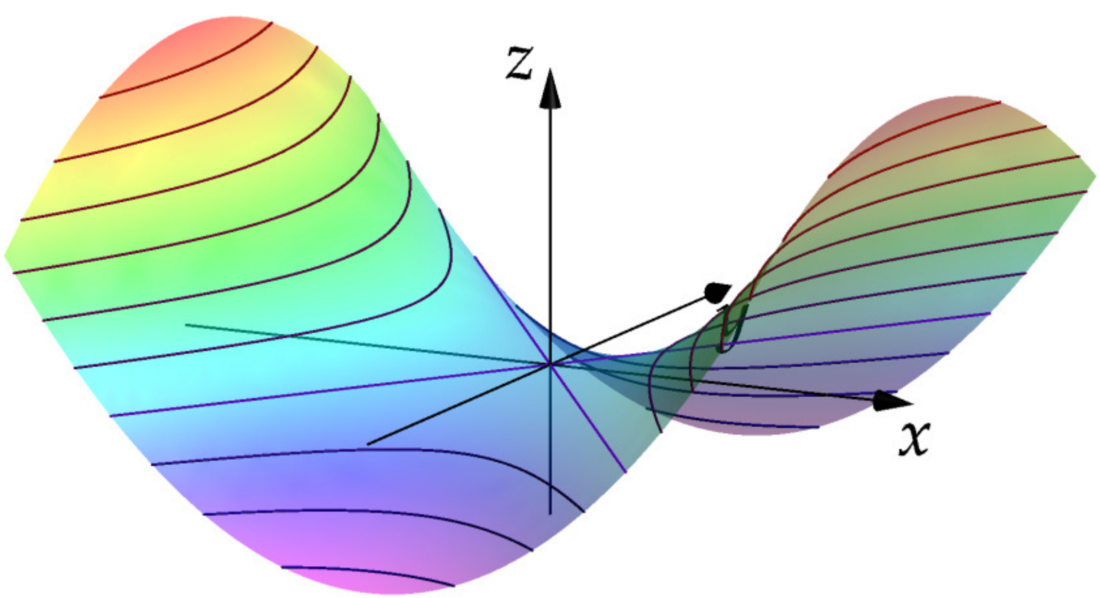
Check: If $F(x,y,z) = z - f(x,y)$, then $\nabla F(1,2) = \mathbf{n}$

The Gradient Vector

$$f(x,y) = \frac{1}{4}x^2 - \frac{1}{8}y^2 \implies \nabla f = \begin{pmatrix} x/2 \\ -y/4 \end{pmatrix}$$



Gradient vector field perpendicular to level curves
Warmer color $\implies$ increased height[1]



[1]This 3D-file is HUGE and might easily crash your computer!