Chapter 14: Written problems

14.1 Functions of Several Variables

1. Find and sketch the domain of the function.

   (a) $f(x,y) = \sqrt{xy}$
   (b) $g(x,y) = \sqrt{x^2 - y^2}$
   (c) $h(x,y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$

2. Sketch some contour lines for each function. Use these to help sketch the graph of the function.

   (a) $f(x,y) = 2 - x$
   (b) $g(x,y) = e^{-y}$
   (c) $h(x,y) = 1 + 2x^2 + 2y^2$
   (d) $j(x,y) = \sqrt{4x^2 + y^2}$

14.2 Limits and Continuity

1. Determine the set of points at which each function is continuous.

   (a) $f(x,y) = \cos\sqrt{1 + x - y}$
   (b) $g(x,y) = \dfrac{e^x + e^y}{e^{xy} - 1}$
   (c) $h(x,y) = \tan^{-1}\big((x^2 + y^2)^{-2}\big)$
   (d) $j(x,y,z) = \sqrt{y - x^2}\,\ln z$

14.3 Partial Derivatives

1. Find the first partial derivatives of each function.

   (a) $f(x,y) = x^4y^3 + 8x^2y$
   (b) $z = \tan xy$
   (c) $w = \dfrac{e^v}{u + v^2}$
   (d) $f(x,y) = x^y$
   (e) $f(x,y,z) = x\sin(y - z)$
   (f) $u = x^{y/z}$
   (g) $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$

2. Find all the second partial derivatives.

   (a) $f(x,y) = \sin^2(mx + ny)$
   (b) $v = \dfrac{xy}{x - y}$
   (c) $v = e^{xe^y}$

3. Show that the Cobb-Douglas production function $P = bL^\alpha K^\beta$ satisfies the equation

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)P$$

4. If a, b, c are the sides of a triangle, and A, B, C the opposite angles, find $\frac{\partial A}{\partial a}, \frac{\partial A}{\partial b}, \frac{\partial A}{\partial c}$ by implicit differentiation of the Law of Cosines.

Everything up to here is on the midterm

14.4 Tangent Planes and Linear Approximations (*after the midterm!*)

1. Find an equation of the tangent plane to the given surface at the specified point.

   (a) $z = 3(x-1)^2 + 2(y+3)^2 + 7, \quad (2, -2, 12)$

   (b) $z = \ln(x - 2y), \quad (3, 1, 0)$

2. Find the linearization of the function $f(x, y) = x^3y^4$ at the point $(1, 1)$.

3. Verify the linear approximation at $(0, 0)$:

$$\sqrt{y + \cos^2 x} \approx 1 + \frac{1}{2}y$$

4. A closed cylindrical can is 10 cm high and 4 cm in diameter. Use differentials to estimate the amount of metal in the can if the top and bottom are 0.1 cm thick and the sides are 0.05 cm thick.

5. Four positive numbers, each at most 50, are rounded to the first decimal place and then multiplied. Use differentials to estimate the maximum possible error in the computed product.

6. The surface S contains the point $P = (2, 1, 3)$, and the curves

$$\mathbf{r}_1(t) = \left\langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \right\rangle, \quad \text{and} \quad \mathbf{r}_2(u) = \left\langle 1 + u^2, 2u^3 - 1, 2u + 1 \right\rangle$$

   both lie on S. Find an equation of the tangent plane to S at P.

14.5 The Chain Rule

1. (a) Use the chain rule to find $\frac{dz}{dt}$ if $z = \cos(x + 4y), \ x = 5t^4$, and $y = \frac{1}{t}$.

   (b) Now substitute x and y into the expression for z and compute the derivative using single-variable calculus.

2. Use the chain rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ if $z = \arcsin(x - y), \ x = s^2 + t^2$, and $y = 1 - 2st$.

   Now substitute and compute as in the previous question.

3. State the chain rule for the derivative $\frac{\partial R}{\partial v}$ if

$$R = f(x, y, z, t), \quad x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w), \quad t = t(u, v, w)$$

4. Suppose that $u = xe^{ty}, \ x = \alpha^2\beta, \ y = \beta^2\gamma$, and $t = \gamma^2\alpha$.

   Use the chain rule to compute the derivatives $\frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta}, \frac{\partial u}{\partial \gamma}$ when $\alpha = -1, \ \beta = 2, \ \gamma = 1$.

5. Wheat production W in a given year depends on the average temperature T and the annual rainfall R. Scientists estimate that the average temperature is rising at a rate of 0.15°C/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels, $\frac{\partial W}{\partial T} = -2$ and $\frac{\partial W}{\partial R} = 8$.

   (a) What is the significance of the signs of these partial derivatives?

   (b) Estimate the current rate of change of wheat production, $\frac{dW}{dt}$.

6. A manufacturer has modeled its yearly production function P (the value of its entire production in millions of dollars) as a Cobb–Douglas function

$$P(L, K) = 1.47L^{0.65}K^{0.35}$$

where L is the number of labor hours (in thousands) and K is the invested capital (in millions of dollars). Suppose that when $L = 30$ and $K = 8$, the labor force is decreasing at a rate of 2000 labor hours per year and capital is increasing at a rate of \$500,000 per year. Find the rate of change of production.

14.6 Directional Derivatives and the Gradient Vector

1. For each function: find the gradient, evaluate it at the point P, then find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

   (a) $f(x, y) = \frac{y^2}{x}$ $\quad P = (1, 2)$ $\quad \mathbf{u} = \frac{1}{3}(2\mathbf{i} + \sqrt{5}\mathbf{j})$

   (b) $f(x, y, z) = y^2 e^{xyz}$ $\quad P = (0, 1, -1)$ $\quad \mathbf{u} = \frac{1}{13}(3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k})$

2. Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

   (a) $f(x, y) = \frac{x}{x^2+y^2}$ $\quad (1, 2)$ $\quad \mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$

   (b) $f(x, y, z) = \sqrt{xyz}$ $\quad (3, 2, 6)$ $\quad \mathbf{v} = -\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

3. The temperature at a point (x, y, z) is given by $T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$ where T is measured in °C and x, y, z in meters.

   (a) Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

   (b) In which direction does the temperature increase fastest?

   (c) Find the maximum rate of increase at P.

4. Find the tangent plane and the normal line to the given surface at the specified point.

   (a) $y = x^2 - z^2$, $\quad (4, 7, 3)$

   (b) $x^4 + y^4 + z^4 = 3x^2y^2z^2$ $\quad (1, 1, 1)$

14.7 Maximum and Minimum Values

1. Find the local maximum and minimum values and saddle point(s) of the function.

   (a) $f(x, y) = xy - 2x - 2y - x^2 - y^2$

   (b) $g(x, y) = xy(1 - x - y)$

   (c) $h(x, y) = y \cos x$

   (d) $j(x, y) = \sin x \sin y$, where $-\pi < x, y < \pi$

2. Find the absolute maximum and minimum values of f on the set D.

   (a) $f(x, y) = x + y - xy$, where D is the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$.

   (b) $f(x, y) = xy^2$, where $D = \{(x, y) \mid x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 3\}$

3. Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

4. Find the dimensions of the rectangular box with largest volume if the total surface area is $64\,\text{cm}^2$.

5. (Hard!) Find an equation for the plane that passes through the point $(1, 2, 3)$ and cuts off the smallest volume in the first/positive octant.

14.8 Lagrange Multipliers

1. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.

   (a) $f(x, y) = 3x + y,$ where $x^2 + y^2 = 10$
   (b) $f(x, y) = e^{xy},$ where $x^3 + y^3 = 16$
   (c) $f(x, y, z) = x^2y^2z^2,$ where $x^2 + y^2 + z^2 = 1$
   (d) $f(x, y, z) = x^4 + y^4 + z^4,$ where $x^2 + y^2 + z^2 = 1$

2. Find the extreme values of the function $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the region given by $x^2 + y^2 \leq 16$

3. We attempt to maximize the function $f(x, y) = 2x + 3y$ subject to the constraint $\sqrt{x} + \sqrt{y} = 5$.

   (a) Try using Lagrange multipliers to solve the problem.
   (b) Evaluate $f(25, 0)$; what do you observe?
   (c) Solve the problem by graphing the constraint function and several level curves of f.
   (d) Explain why the method of Lagrange multipliers fails to solve the problem.
   (e) What is the significance of $f(9, 4)$?