Chapter 15: Written problems

15.1 Double Integrals Over Rectangles

1. Evaluate the double integral $\iint_R (5-x)\,\mathrm{d}A$ where $R = \{(x,y) : 0 \le x \le 5,\ 0 \le y \le 3\}$ by identifying it as the volume of a solid.

2. The integral $\iint_R \sqrt{9-y^2}\,\mathrm{d}A$, where $R = [0,4] \times [0,2]$, represents the volume of a solid. Sketch and describe the solid. Don't evaluate its volume.

3. If $f(x,y) = k$ is a constant function and $R = [a,b] \times [c,d]$, use the Riemann sum definition to show that

$$\iint_R k\,\mathrm{d}A = k(b-a)(d-c)$$

4. Calculate the iterated integral.

   (a) $\displaystyle\int_0^1 \int_1^2 (4x^2 - 9x^2y^2)\,\mathrm{d}y\,\mathrm{d}x$ (b) $\displaystyle\int_1^3 \int_1^5 \frac{\ln y}{xy}\,\mathrm{d}y\,\mathrm{d}x$ (c) $\displaystyle\int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$

5. Calculate the double integral.

   (a) $\displaystyle\iint_R \sin(x-y)\,\mathrm{d}A$ where $R = \{(x,y) : 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{2}\}$

   (b) $\displaystyle\iint_R \frac{x}{1+xy}\,\mathrm{d}A$ where $R = [0,1] \times [0,1]$

6. Find the volume of the solid that lies under the hyperbolic paraboloid $z = 3y^2 - x^2 + 2$ and above the rectangle $R = [-1,1] \times [1,2]$.

7. Find the volume of the solid enclosed by the surface $z = 1 + e^x \sin y$ and the planes $x = \pm 1$, $y = 0$, $y = \pi$, and $z = 0$.

8. Use symmetry to evaluate the double integral

$$\iint_R (1 + x^2 \sin y + y^2 \sin x)\,\mathrm{d}A, \qquad R = [-\pi,\pi] \times [-\pi,\pi]$$

15.2 Double Integrals over General Regions

1. Evaluate the integral $\displaystyle\int_0^2 \int_y^{2y} xy\,\mathrm{d}x\mathrm{d}y$.

2. Evaluate the integral $\displaystyle\iint_D \frac{y}{x^5+1}\,\mathrm{d}A$ where $D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x^2\}$.

3. Evaluate the integral $\displaystyle\iint_D x^3\,\mathrm{d}A$ where $D = \{(x,y) : 1 \le x \le e,\ 0 \le y \le \ln x\}$.

4. Express D as a region of type I and also as a region of type II. Then evaluate the double integral in two ways.

$$\iint_D xy\,\mathrm{d}A, \quad D \text{ is enclosed by the curves } y = x^2,\ y = 3x$$

5. Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

$$\iint_D y^2 e^{xy}\,\mathrm{d}A, \quad D \text{ is bounded by } y = x,\ y = 4,\ x = 0$$

6. Find the volume of the given solid.
   (a) Under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$.
   (b) Enclosed by the paraboloid $z = x^2 + 3y^2$ and the planes $x = 0$, $y = 1$, $y = x$, $z = 0$.
   (c) Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant.

7. Sketch the region of integration and change the order of integration.

   (a) $\displaystyle\int_0^2 \int_{x^2}^4 f(x,y)\,\mathrm{d}y\,\mathrm{d}x$ (b) $\displaystyle\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

8. Evaluate the integral by reversing the order of integration.

   (a) $\displaystyle\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos(x^2)\,\mathrm{d}x\,\mathrm{d}y$ (b) $\displaystyle\int_0^1 \int_x^1 e^{x/y}\,\mathrm{d}y\,\mathrm{d}x$

15.3 Double Integrals in Polar Co-ordinates

1. Sketch the region whose area is given by the integral $\int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r\,\mathrm{d}r\mathrm{d}\theta$ and evaluate the area.

2. By changing to polar co-ordinates, evaluate the integral $\iint_R \frac{y^2}{x^2+y^2}\,\mathrm{d}A$, where R is the region lying between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ for $0 < a < b$.

3. By changing to polar co-ordinates, evaluate the integral $\iint_D \cos\sqrt{x^2+y^2}\,\mathrm{d}A$, where D is the disk with center the origin and radius 2.

4. Use polar co-ordinates to find the volume of the solid lying below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy-plane.

5. Use polar co-ordinates to find the volume of the solid bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

6. (a) A cylindrical drill of radius r_1 is used to bore a hole through the center of a sphere of radius r_2. Find the volume of the ring-shaped solid that remains.
   (b) Express the volume in part (a) in terms of the height h of the ring. Notice that the volume depends only on h, not on r_1 or r_2.

7. Let D be the disk with center the origin and radius a. What is the average distance from points in D to the origin?