Math 2D: Summer 2011 Final Exam

Total 100 marks: marks per question are in brackets. Remember to show working for full credit.

1. Suppose that the function $f(x,y)$ is *continuous* at (a,b). Let k,l be constants. What can you say about the limits
$$\lim_{t\to 0} f(a+kt, b+lt),$$
for all k,l? (5)

2. Consider the function $f(x,y) = 100x + 50y - 3x^2 - 2y^2 - 2xy$.
   (a) Calculate the partial derivatives of f with respect to x, y. (4)
   (b) Show that the tangent plane to $z = f(x,y)$ at $(x,y) = (10,10)$ is given by (5)
$$z = 20x - 10y + 700.$$
   (c) Suppose that the function f represents the profit made by a company when it sells x units of product X and y units of product Y. Suppose that the company's current weekly sales are 10 of A and 10 of B. If sales of product X increase to 10.1 units per week, *approximately* how many units of Y will the company need to sell in order to keep profits constant? (6)

3. (a) Find the critical points of the function $f(x,y) = x^3 + 3xy^2 + 3x^2y + 3y$. (7)
   (b) Show that the second derivative test fails to provide any information about these critical points. Can you identify the nature of the critical points? Give reasons for your answer. (8)

4. For the following integral, sketch the region of integration and evaluate by changing the order of integration: (10)
$$\int_0^4 \int_{\sqrt{x}}^2 y \exp(y^4)\, \mathrm{d}y\, \mathrm{d}x.$$

5. Use polar co-ordinates to find the volume of the region inside both the cylinder $x^2 + y^2 = 9$ and the ellipsoid $x^2 + y^2 + 4z^2 = 16$. (10)

6. Consider the problem of finding the maximum and minimum values of the function $f(x,y,z) = x + 2y - 3z$ subject to the constraint $4x^2 + 2xy + 9y^2 + z^2 = 960$.
   (a) You are given that the points satisfying the constraint lie on an ellipsoid. Make a sketch to show why the problem has a maximum and a minimum value, and how many of each there are. (5)
   (b) Solve the problem using Lagrangian multipliers. (10)

7. The height of a mountain is modeled by the equation $h(x,y) = 11\exp\left(\frac{-x^2-2y^2}{33}\right)$ where x, y, h are measured in 1000s of ft.
   (a) Find the directional derivative of the function at $(5,2)$ in the direction $\frac{1}{5}(4\mathbf{i} + 3\mathbf{j})$. (7)
   (b) A hiker follows a path parameterized by $x = t, y = t^{1/2}$ from the summit. Calculate the slope of the path the hiker walks as a function of t. (8)

8. A large peach is modeled by the equation $\rho = 1 - \cos\phi$ ($0 \leq \phi \leq \pi$) in spherical polar co-ordinates (a vertical cross-section is shown below — rotate this around the z-axis). If ρ is measured in inches, find the volume of the peach. (15)