14.6 Directional Derivatives and the Gradient Vector

8, 10  (a) Find the gradient of \( f \).
(b) Evaluate the gradient at the point \( P \).
(c) Find the rate of change of \( f \) at \( P \) in the direction of the vector \( u \).

\[ 8 \quad f(x, y) = \frac{y^2}{x}, \quad P(1, 2), \quad u = \frac{1}{3}(2i + \sqrt{5}j) \]

\[ 10 \quad f(x, y, z) = y^2e^{xyz}, \quad P(0, 1, -1), \quad u = \frac{1}{13}(3i + 4j + 12k) \]

12, 16 Find the directional derivative of the function at the given point in the direction of the vector \( v \).

\[ 12 \quad f(x, y) = \frac{x}{x^2 + y^2}, \quad (1, 2), \quad v = 3i + 5j \]

\[ 16 \quad f(x, y, z) = \sqrt{xyz}, \quad (3, 2, 6), \quad v = -i - 2j + 2k \]

32 * The temperature at a point \((x, y, z)\) is given by

\[ T(x, y, z) = 200e^{-x^2 - 3y^2 - 9z^2} \]

where \( T \) is measured in \(^\circ\)C and \( x, y, z \) in meters.

(a) Find the rate of change of temperature at the point \( P(2, -1, 2) \) in the direction toward the point \((3, -3, 3)\).
(b) In which direction does the temperature increase fastest?
(c) Find the maximum rate of increase at \( P \).

42, 46 Find the equations of (a) the tangent plane and (b) the normal line to the given surface at the specified point.

\[ 42 \quad y = x^2 - z^2, \quad (4, 7, 3) \]

\[ 46 \quad x^4 + y^4 + z^4 = 3x^2y^2z^2, \quad (1, 1, 1) \]

14.7 Maximum and Minimum Values

6–18 Find the local maximum and minimum values and saddle point(s) of the function.

\[ 6 \quad f(x, y) = xy - 2x - 2y - x^2 - y^2 \]

\[ 10 \quad f(x, y) = xy(1 - x - y) \]

\[ 14 \quad f(x, y) = y \cos x \]

\[ 18 \quad f(x, y) = \sin x \sin y, \quad -\pi < x < \pi, \quad -\pi < y < \pi \]

30, 34 Find the absolute maximum and minimum values of \( f \) on the set \( D \).
30 * $f(x, y) = x + y - xy$, $D$ is the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$.

34 $f(x, y) = xy^2$, $D = \{(x, y) \mid x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$

42 Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

48 Find the dimensions of the rectangular box with largest volume if the total surface area is 64 cm$^2$.

56 * Find an equation of the plane that passes through the point $(1, 2, 3)$ and cuts off the smallest volume in the first octant.