16 Vector Calculus

16.1 Vector Fields

Definition. A *vector field* in the plane is a function $\mathbf{F}(x, y)$ from a domain $D \subseteq \mathbb{R}^2$ to the set of vectors in the plane. We write

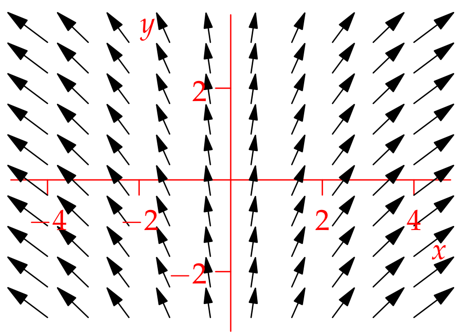
$$\mathbf{F}(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j} = \langle f_1(x,y), f_2(x,y) \rangle$$

f_1, f_2 are the *scalar components* of $\mathbf{F}$.

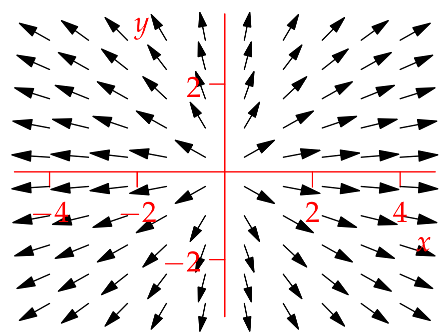
Notation and terminology

- Vector fields are typed in **bold**. They can be hand-written underlined $\underline{F}$ or with an arrow $\vec{F}$.
- The *magnitude* of a vector field is the scalar function $|\mathbf{F}(x,y)|$ returning the *length* of the vector $\mathbf{F}(x,y)$ at each point (x,y).
- The radial vector field $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ has the common short-hand $\mathbf{r}$. This is congruent with the fact that its magnitude $|\mathbf{r}|$ is the scalar quantity $r = \sqrt{x^2 + y^2}$. Some authors use $\mathbf{x}$ for the same field.

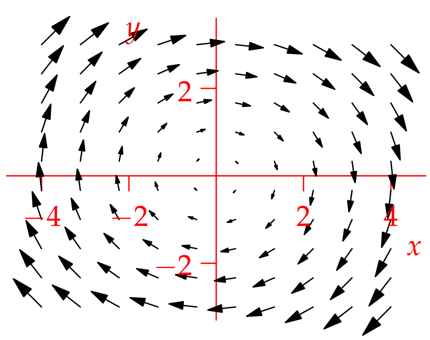
Examples A vector field $\mathbf{F}(x,y)$ can be sketched as a family of arrows. Place the tail of each arrow at (x,y) and the nose at $\big(x + f_1(x,y), y + f_2(x,y)\big)$ (a computer typically shrinks vector lengths for clarity).



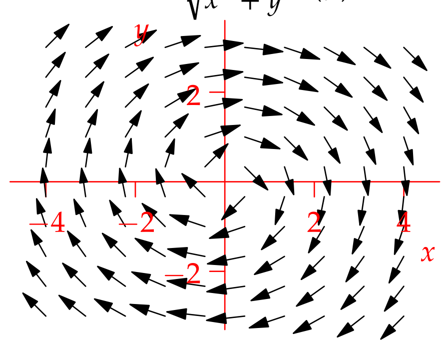
1. $\mathbf{F}(x,y) = \begin{pmatrix} x/4 \\ 1 \end{pmatrix}$



2. $\mathbf{F}(x,y) = \dfrac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{r}\mathbf{r}$



3. $\mathbf{F}(x,y) = \begin{pmatrix} y \\ -x \end{pmatrix}$



4. $\mathbf{F}(x,y) = \dfrac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} y \\ -x \end{pmatrix}$

Note that examples 2. and 4. have domain $D = \mathbb{R}^2 \setminus \{(0,0)\}$ (not including the origin).

In real life, vector fields typically represent one of two physical ideas:

> ***Velocity Field*** A particle at location $\mathbf{r}$ has *velocity* $\mathbf{F}(\mathbf{r})$.
>
> ***Force Field*** A particle at location $\mathbf{r}$ experiences a *force* $\mathbf{F}(\mathbf{r})$.

A map of ocean currents or wind patterns might be interpreted as either: the velocity of the water/wind at a point or the force of the water/wind on an object.

Vector Fields in 3D

These are identical, just with an additional component.

Definition. A *vector field* in three dimensions is a function $\mathbf{F}(x,y,z)$ from a domain $D \subseteq \mathbb{R}^3$ to the set of vectors in 3-space. We write

$$\begin{aligned}\mathbf{F}(x,y,z) &= \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix} = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k} \\ &= \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle\end{aligned}$$

Plotting vector fields in three dimensions is not advisable by hand! To help visualize a 3D vector field, start by covering up, say, the vertical component: if you know what the field is doing horizontally and vertically, you can put these together in your mind.

Just as in 2D, we often write $\mathbf{r} = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Be careful, since this last is what we called ρ when computing spherical polar integrals.

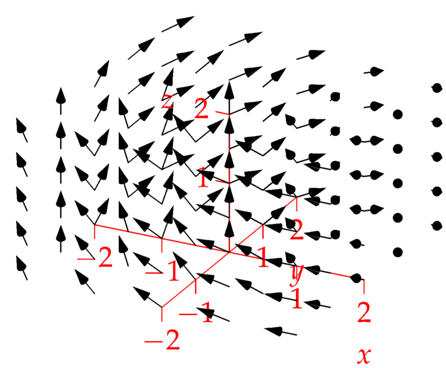
Examples 1. $\mathbf{F}(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+1}} \begin{pmatrix} y \\ -x \\ 1 \end{pmatrix}$

Unit vector field: $|\mathbf{F}| = 1$.

Horizontal field rotates clockwise (compare example 3, 4 above).

Vertical part ($\mathbf{k}$) points upwards.

If a feather were dropped into this vector field, it would follow an ascending *helix.*

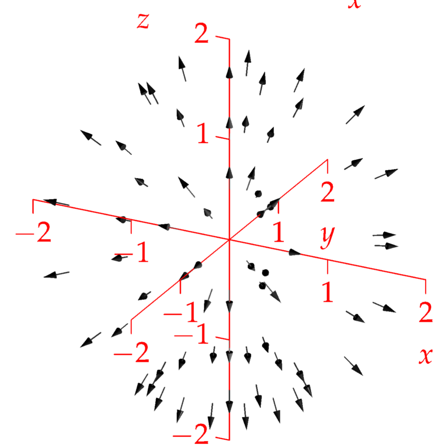


2. $\mathbf{F}(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dfrac{1}{r}\mathbf{r}$

Also a unit vector field.

All vectors point away from origin.

Domain excludes the origin.



Conservative Vector Fields

Definition. The *gradient field* of a scalar function f (the *potential function*) is the vector field

$$\mathbf{F} = \nabla f = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix} \quad \text{depending on dimension}$$

If $\mathbf{F} = \nabla f$ for some f we say that $\mathbf{F}$ is *conservative.*

Physicists often use ϕ for potential functions, and typically choose a potential function to be the negative of ours $\mathbf{F} = -\nabla f = -\nabla \phi$ (a good reason for this will be given later).

Examples 1. $\nabla(xy) = \begin{pmatrix} y \\ x \end{pmatrix}$

2. $\nabla(x^2 \cos y - \sin x) = \begin{pmatrix} 2x \cos y - \cos x \\ -x^2 \sin y \end{pmatrix}$

3. Since $\frac{\partial}{\partial x}(x^2 + y^2)^{1/2} = 2x \cdot \frac{1}{2}(x^2 + y^2)^{-1/2}$, etc., we have

$$\nabla\big((x^2 + y^2)^{1/2}\big) = (x^2 + y^2)^{-1/2} \begin{pmatrix} x \\ y \end{pmatrix}$$

This can be written $\nabla r = \frac{1}{r}\mathbf{r}$ in polar co-ordinates.

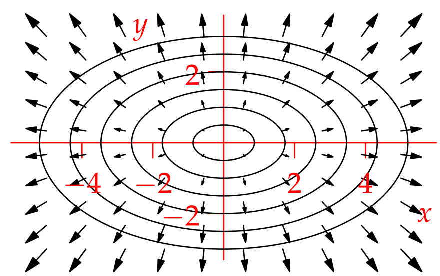
4. In 3-dimensions

$$\nabla(x^2 + y^2 + z^2)^{-1/2} = -(x^2 + y^2 + z^2)^{-3/2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which can be written $\nabla r^{-1} = -\frac{1}{r^3}\mathbf{r}$.

Level Curves/Surfaces of Potential Functions Recall that the gradient vector ∇f points in the direction of greatest increase of f, orthogonally to the level curves/surfaces of f.

Example the gradient field $\nabla(3x^2 + y^2) = \begin{pmatrix} 6x \\ 2y \end{pmatrix}$ is orthogonal to the ellipses with equations $3x^2 + y^2 = C$ ($C > 0$ constant).



Are all vector fields conservative? No: in fact most are not! Conservative vector fields are very rare, though they are very common in important theories from Physics.

Example $\mathbf{F} = \begin{pmatrix} x \\ x \end{pmatrix}$ is not conservative. If it were, there would exist a function f for which $f_x = x = f_y$. We can attempt to 'partially integrate' both of these equations. When integrating with respect to x, the 'constant of integration' $g(y)$ is an arbitrary function of y. Similarly when integrating with respect to y we get a constant of integration $h(x)$.

$$f_x = x \implies f(x,y) = \frac{1}{2}x^2 + g(y)$$
$$f_y = x \implies f(x,y) = xy + h(x)$$

It is impossible to choose functions $g(y), h(x)$ so that the above expressions for f are *both* true, whence $\mathbf{F}$ is non-conservative.

Later we will see a much easier way to show that a vector field is non-conservative.