16 Vector Calculus

16.1 Vector Fields

Definition. *A vector field in the plane is a function* $\mathbf{F}(x, y)$ *from a domain* $D \subseteq \mathbb{R}^2$ *to the set of vectors in the plane. We write*

$$\mathbf{F}(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = f_1(x,y)\mathbf{i} + f_2(x,y)\mathbf{j} = \langle f_1(x,y), f_2(x,y) \rangle$$

f_1, f_2 *are the scalar components of* $\mathbf{F}$

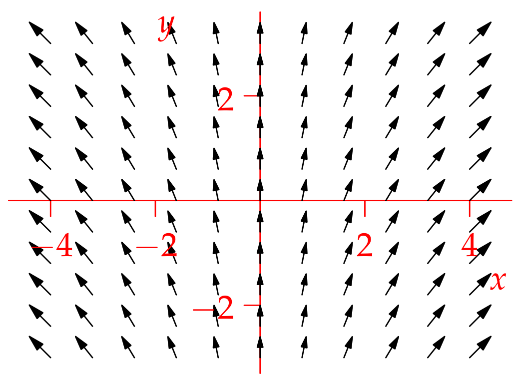
A vector field $\mathbf{F}(x, y)$ consists of a set of arrows: one for every pair (x, y). The tail of each arrow is at the point (x, y) and the nose at $\big(x + f_1(x, y), y + f_2(x, y)\big)$.
When plotting, choose enough points (x, y) to make the overall picture clear.

Notation and terminology

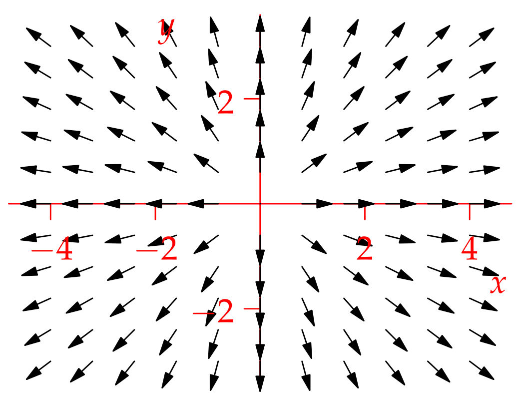
- Vector fields, like vectors, are typed in **bold**. They can be hand-written either underlined $\underline{\text{F}}$ (preferred), or with an arrow $\vec{\text{F}}$.
- The *magnitude* of a vector field is the scalar function $|\mathbf{F}(x, y)|$ returning the *length* of the vector $\mathbf{F}(x, y)$ at each point (x, y).
- The radial vector field $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ has the common short-hand $\mathbf{r}$. This is congruent with the fact that its magnitude $|\mathbf{r}|$ is the scalar quantity $r = \sqrt{x^2 + y^2}$. Some authors use $\mathbf{x}$ for the same field.

Examples For the following, note that the computer typically shrinks the lengths of vectors in order to draw more for clarity.

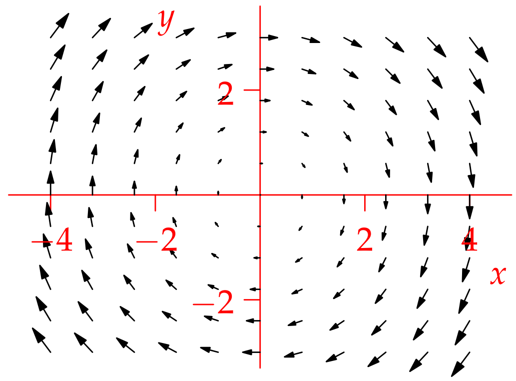
1. $\mathbf{F}(x,y) = \begin{pmatrix} x/4 \\ 1 \end{pmatrix}$



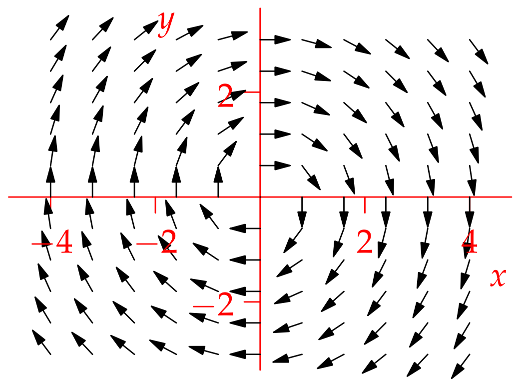
2. $\mathbf{F}(x,y) = \dfrac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{r}\mathbf{r}$



3. $\mathbf{F}(x,y) = \begin{pmatrix} y \\ -x \end{pmatrix}$



4. $\mathbf{F}(x,y) = \dfrac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} y \\ -x \end{pmatrix}$



Note that examples 2. and 4. have domain $D = \mathbb{R}^2 \setminus \{(0,0)\}$.

Vector fields in real life

Vector fields typically represent one of two physical ideas:

Velocity Fields A particle at location $\mathbf{r}$ has *velocity* $\mathbf{F}(\mathbf{r})$

Force Fields A particle at location $\mathbf{r}$ experiences a *force* $\mathbf{F}(\mathbf{r})$

A map of wind patterns could be interpreted as either: each arrow represents

1. The wind velocity at a point; how fast the air is moving
2. The force of the wind on a stationary observer

Examples Here are three examples of vector fields and associated data being used to usefully summarize data

Ocean Currents
Wind Patterns
Animated Wind Patterns

Vector Fields in 3D

Definition. *A* vector field *in three dimensions is a function* $\mathbf{F}(x,y,z)$ *from a domain* $D \subseteq \mathbb{R}^3$ *to the set of vectors in 3-space. We write*

$$\mathbf{F}(x,y,z) = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix} = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$$
$$= \langle f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) \rangle$$

where f_1, f_2, f_3 *are the scalar components of* $\mathbf{F}$*.*

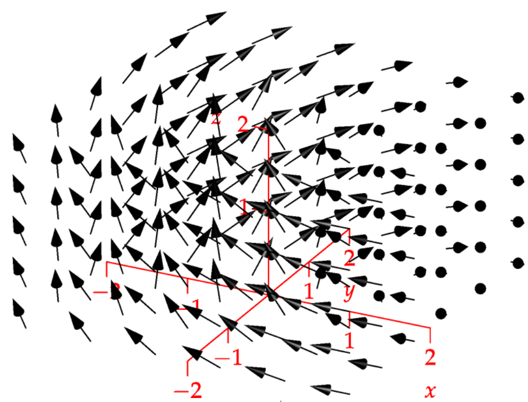
Plotting vector fields in three dimensions is not advisable by hand — use a computer package! When you have to visualize a 3D vector field, see if you can build up your understanding by covering up, say, the vertical component: if you know what the field is doing horizontally, then vertically, you can put these together in your mind.

Notation: Just as in 2D, we have the short-hands $\mathbf{r} = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right)$ and $r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$: this last is what we called ρ when computing spherical polar intergals. In vector calculus, r usually means $|\mathbf{r}|$, regardless of the dimension.

Examples

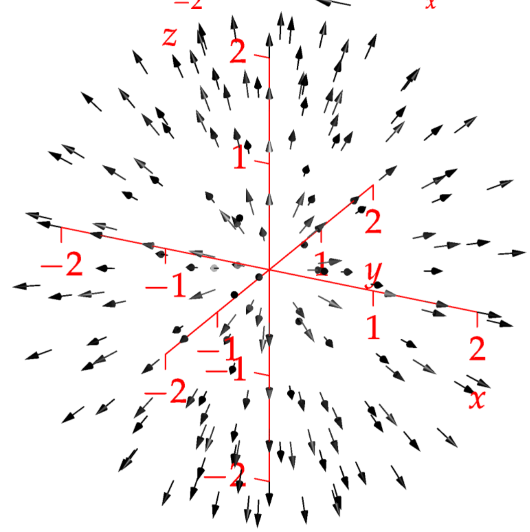
1. $\mathbf{F}(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+1}} \begin{pmatrix} y \\ -x \\ 1 \end{pmatrix}$

   Unit vector field.
   Horizontal ($\mathbf{i}, \mathbf{j}$) parts rotating clockwise (compare with examples 3, 4 above).
   Vertical ($\mathbf{k}$) part pointing upwards.
   If a feather was dropped into this vector field, it would follow an ascending *helix*.



2. $\mathbf{F}(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dfrac{1}{r}\mathbf{r}$

   Also a unit vector field.
   All vectors point away from origin, length 1.
   Domain excludes origin.



Gradient Fields

Definition. *The* gradient field *of a scalar function* $f(x,y)$ *is the vector field*

$$\mathbf{F}(x,y) = \nabla f(x,y) = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix}$$

f *is a* potential function *for* $\mathbf{F}$.
If $\mathbf{F} = \nabla f$ *for some* f *we say that* $\mathbf{F}$ *is* conservative.

The definition is similar in 3D: $\nabla f(x,y,z) = \begin{pmatrix} f_x(x,y,z) \\ f_y(x,y,z) \\ f_z(x,y,z) \end{pmatrix}$

Notation ϕ is often used in Physics for potential functions.
Physicists also typically choose a potential function to be the negative of ours $\mathbf{F} = -\nabla f = -\nabla \phi$ (a good reason will be given later).

Examples

1. $\nabla(xy) = \begin{pmatrix} y \\ x \end{pmatrix}$
2. $\nabla(x^2 \cos y - \sin x) = \begin{pmatrix} 2x\cos y - \cos x \\ -x^2 \sin y \end{pmatrix}$
3. Since $\frac{\partial}{\partial x}(x^2+y^2)^{1/2} = 2x \cdot \frac{1}{2}(x^2+y^2)^{-1/2}$, etc., we have

   $$\nabla((x^2+y^2)^{1/2}) = (x^2+y^2)^{-1/2} \begin{pmatrix} x \\ y \end{pmatrix}$$

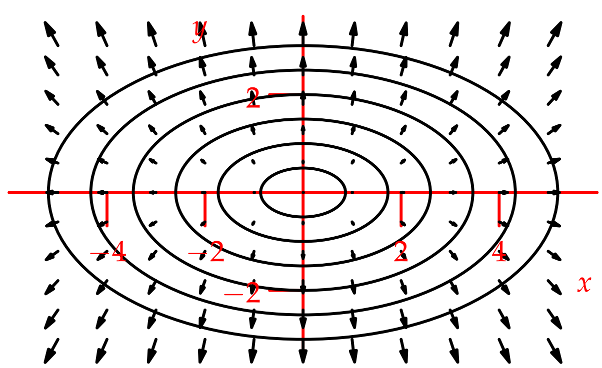
   This can be written $\nabla r = \frac{1}{r}\mathbf{r}$ in polar co-ordinates.
4. In 3-dimensions

   $$\nabla(x^2+y^2+z^2)^{-1/2} = -(x^2+y^2+z^2)^{-3/2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

   which can be written $\nabla r^{-1} = -\frac{1}{r^3}\mathbf{r}$ (or $\nabla \rho^{-1} = -\frac{1}{\rho^3}\mathbf{r}$ if you prefer).

Level Curves/Surfaces Recall that the gradient vector ∇f points in the direction of greatest increase of f. Otherwise said, ∇f points orthogonally to the level curves of f (level surfaces in 3D).

Example The gradient field $\nabla(3x^2+y^2) = \begin{pmatrix} 6x \\ 2y \end{pmatrix}$ is orthogonal to the ellipses with equations $3x^2 + y^2 = C$ ($C > 0$ constant).



Are all vector fields conservative? No: in fact most are not! We will see that potential functions play a similar role to that of anti-derivatives with respect to the Fundamental Theorem in single variable calculus. While any continuous function has an anti-derivative, most continuous vector fields do not have a potential function. Conservative vector fields are, in a probabilistic sense, very rare. They are however, very common in important theories from Physics.

Example $\mathbf{F} = \begin{pmatrix} x \\ x \end{pmatrix}$ is not conservative

If it were conservative, there would exist a function f for which $f_x = x = f_y$
We can attempt to 'partially integrate' both of these equations. When integrating with respect to x, the 'constant of integration' $g(y)$ is an arbitrary function: the most general expression whose x-derivative is zero.

$$f_x = x \implies f(x,y) = \frac{1}{2}x^2 + g(y)$$
$$f_y = x \implies f(x,y) = xy + h(x)$$

for some arbitrary functions g, h.
If f is to exist, and $\mathbf{F}$ is to be conservative, we must be able to choose funcrions $g(y), h(x)$ so that the above expressions for f are *both* true. This is clearly impossible, whence $\mathbf{F}$ is non-conservative.

Later we will see a much easier way to show that a vector field is non-conservative.