16.2 Line Integrals

Definition. Let f be a continuous scalar function and C a curve in 2-space parametrized by a function

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, \quad a \le t \le b$$

where $x(t), y(t)$ have continuous first derivatives. We define two types of line integral.

1. The *line integral of f along C* ***with respect to arc-length*** is

   $$\int_C f\,\mathrm{d}s := \int_a^b f\big(\mathbf{r}(t)\big)\,|\mathbf{r}'(t)|\,\mathrm{d}t = \int_a^b f\big(x(t), y(t)\big)\sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t$$

   The *arc-length*[1] of the curve is the integral $\int_C \mathrm{d}s = \int_a^b f\,|\mathbf{r}'(t)|\,\mathrm{d}t = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t$.

2. The *line integral of f along C* ***with respect to*** x is

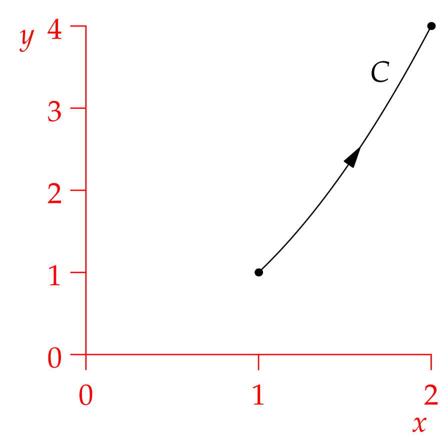
   $$\int_C f\,\mathrm{d}x := \int_a^b f\big(x(t), y(t)\big)x'(t)\,\mathrm{d}t$$

   The line integral with respect to y is similar.

The definitions are identical in 3-dimensions, except that f is now a function of three variables: e.g.

$$\int_C f\,\mathrm{d}s := \int_a^b f\big(x(t), y(t), z(t)\big)\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,\mathrm{d}t$$

Example 1 If C is the parabola $y = x^2$ for $1 \le x \le 2$, compute $\int_C \frac{y}{x}\,\mathrm{d}s$ and $\int_C 5y^2\,\mathrm{d}x + 2\,\mathrm{d}y$.



First parametrize the curve. The obvious choice is plainly

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

from which $\mathrm{d}s = \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t = \sqrt{1+4t^2}\,\mathrm{d}t$. We therefore have,

$$\begin{aligned}\int_C \frac{y}{x}\mathrm{d}s &= \int_1^2 \frac{t^2}{t}\sqrt{1+4t^2}\,\mathrm{d}t = \int_1^2 t(1+4t^2)^{1/2}\,\mathrm{d}t \\ &= \frac{1}{3/2}\cdot\frac{1}{8}[1+4t^2]^{3/2}\Big|_1^2 = \frac{1}{12}(17^{3/2} - 5^{3/2})\end{aligned}$$

$$\int_C 5y^2\,\mathrm{d}x + 2\,\mathrm{d}y = \int_1^2 5t^4\,\mathrm{d}t + 2\cdot 2t\,\mathrm{d}t = t^5 + 2t^2\Big|_1^2 = 37$$

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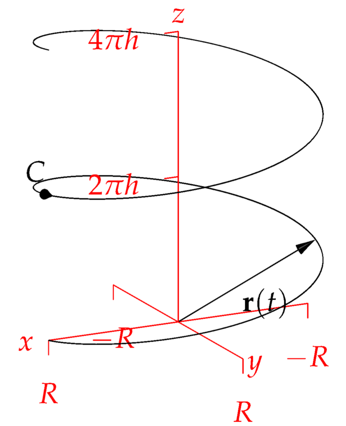
[1]The letter s usually indicates *distance* or *displacement* in Physics. If, at time t, a particle has traveled a distance $s(t)$ along a curve C parametrized by $\mathbf{r}(t)$, then its speed is $\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{r}'(t)|$. Integrating both sides with respect to t must recover the arc-length of the curve:

$$\text{Arc-length} = \int_a^b \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t = \int_a^b |\mathbf{r}'(t)|\,\mathrm{d}t = \int_C 1\,\mathrm{d}s$$

Example 2 A length of wire is shaped as a helix parametrized by

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} R\cos t \\ R\sin t \\ ht \end{pmatrix}, \quad 0 \le t \le 4\pi$$



where R and h are positive constants. Find the mass of the wire if it has: (a) constant density ρ (mass per unit length), and (b) density $\rho(x,y,z) = z(4\pi h - z)$ ((thinner at the ends of the wire).

Since mass is the integral of density, we obtain

$$\begin{aligned}\int_C \rho\big(\mathbf{r}(t)\big)\,\mathrm{d}s &= \int_0^{4\pi} \rho(\mathbf{r}(t))\sqrt{R^2\sin^2 t + R^2\cos^2 t + h^2}\,\mathrm{d}t \\ &= \sqrt{R^2+h^2}\int_0^{4\pi}\rho\big(\mathbf{r}(t)\big)\,\mathrm{d}t\end{aligned}$$

(a) If ρ is constant this evaluates to $4\pi\rho\sqrt{R^2+h^2}$.

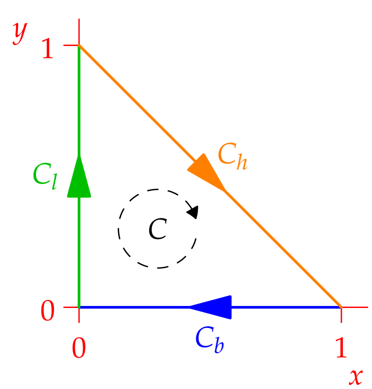
(b) For the tapered wire, the mass is

$$\begin{aligned}\int_C \rho\big(\mathbf{r}(t)\big)\,\mathrm{d}s &= \sqrt{R^2+h^2}\int_0^{4\pi} z(4\pi h - z)\,\mathrm{d}t = \sqrt{R^2+h^2}\int_0^{4\pi} ht\,(4\pi h - ht)\,\mathrm{d}t \\ &= h^2\sqrt{R^2+h^2}\int_0^{4\pi} 4\pi t - t^2\,\mathrm{d}t = h^2\sqrt{R^2+h^2}\cdot\frac{1}{6}(4\pi)^3 \\ &= \frac{32\pi^3}{3}h^2\sqrt{R^2+h^2}\end{aligned}$$

Example 3: A piecewise smooth curve To compute $\int_C x^2(y+1)\,\mathrm{d}s$ over the triangle C shown, split the curve, and therefore the integral, into three pieces.

Base $\mathbf{r}_b(t) = \binom{1-t}{0}$, $0 \le t \le 1$ yields

$$\int_{C_b} f\,\mathrm{d}s = \int_0^1 (1-t)^2\sqrt{(-1)^2+0^2}\,\mathrm{d}t = \frac{1}{3}$$



Left $f = 0 \implies \int_{C_l} f\,\mathrm{d}s = 0$

Hypotenuse $\mathbf{r}_h(t) = \binom{t}{1-t}$, $0 \le t \le 1$ gives

$$\int_{C_h} f\,\mathrm{d}s = \int_0^1 t^2(1-t+1)\sqrt{1^2+1^2}\,\mathrm{d}t = \sqrt{2}\int_0^1 2t^2 - t^3\,\mathrm{d}t = \frac{5\sqrt{2}}{12}$$

The total integral is therefore $\int_C x^2(y+1)\,\mathrm{d}s = \frac{1}{3} + \frac{5\sqrt{2}}{12}$.

Interpretation: Riemann Sums and Averages

Fix $n \in \mathbb{N}$, suppose that $(x_0, y_0), \dots, (x_n, y_n)$ are a sequence of equally-spaced points along the curve C separated by distances

$$\Delta s_i = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \qquad (\Delta s_1 = \Delta s_2 = \cdots = \Delta s_n!)$$

The line integral of f along C is then the limit of a sequence of Riemann sums:

$$\begin{aligned}\int_C f\,\mathrm{d}s &= \lim_{n\to\infty} \sum_{i=1}^n f(x_i, y_i)\Delta s_i = \lim_{n\to\infty} \sum_{i=1}^n \frac{f(x_i, y_i)}{n} n\Delta s_i \\ &= \left(\lim_{n\to\infty} \sum_{i=1}^n \frac{f(x_i, y_i)}{n}\right)\left(\lim_{n\to\infty} n\Delta s_1\right) = \left(\lim_{n\to\infty} \sum_{i=1}^n \frac{f(x_i, y_i)}{n}\right)(\text{arc-length of } C)\end{aligned}$$

The first term is the average value of the function f along the curve C.
Provided the x co-ordinate of a point on C always moves in the same direction, we may interpret

$$\int_C f\,\mathrm{d}x = (\text{average value of } f \text{ along } C) \cdot (\text{change in } x \text{ along } C)$$

However, if the curve C starts to double back on itself, this interpretation doesn't hold.

Example A hiker climbs a mountain pass modeled by the equation $y = 4 - \frac{1}{9}x^2$ from $x = -3$ to $x = 3$, with distance measured in kilometers. If the air pressure at y km above sea level is modeled by

$$P = 100\left(1 - \frac{y}{50}\right)^{5.3} \text{ kPa}$$

Express the average air pressure along the trail using line integrals.
Parametrize the trail by $\mathbf{r}(t) = \left(\begin{smallmatrix} 3t \\ 4-t^2 \end{smallmatrix}\right)$ for $-1 \le t \le 1$. The average air pressure is then

$$P_{\text{av}} = \frac{\int_{-1}^1 P\big(\mathbf{r}(t)\big)\,|\mathbf{r}'(t)|\,\mathrm{d}t}{\int_{-1}^1 |\mathbf{r}'(t)|\,\mathrm{d}t} = \frac{1}{\int_{-1}^1 \sqrt{9+4t^2}\,\mathrm{d}t} \int_{-1}^1 100\left(1 - \frac{4-t^2}{50}\right)^{5.3} \sqrt{9+4t^2}\,\mathrm{d}t$$

The numerator cannot be evaluated exactly, but with the assistance of a computer we estimate

$$P_{\text{av}} = 66.95\,\text{kPa, to 2 d.p.}$$

On average, the hiker will have approximate $\frac{2}{3}$ of the oxygen available at sea-level.

Line Integrals though Vector Fields: Work

Line integrals with respect to the co-ordinate functions x, y, z appear very often in physical applications.

Definition. Suppose $\mathbf{F}(x, y) = \left(\begin{smallmatrix} P(x,y) \\ Q(x,y) \end{smallmatrix}\right)$ is a vector field and C a curve whose domain lies inside that of $\mathbf{F}$. The *line integral of* $\mathbf{F}$ *along* C is the integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y \qquad (\text{add } R\,\mathrm{d}z \text{ in three dimensions})$$

This is also known as the *work* done by the field to a particle moving along C. Strictly this terminology should only be used when $\mathbf{F}$ is a *force field* (measured in Newtons).

Examples 1. A particle moves once around C_1, the circle $x^2+y^2 = R^2$ of radius R. We may parametrize this $\mathbf{r}(t) = \left(\begin{smallmatrix} R\cos t \\ R\sin t \end{smallmatrix}\right)$, from which $\mathrm{d}\mathbf{r} = R\left(\begin{smallmatrix} -\sin t \\ \cos t \end{smallmatrix}\right)\mathrm{d}t$.

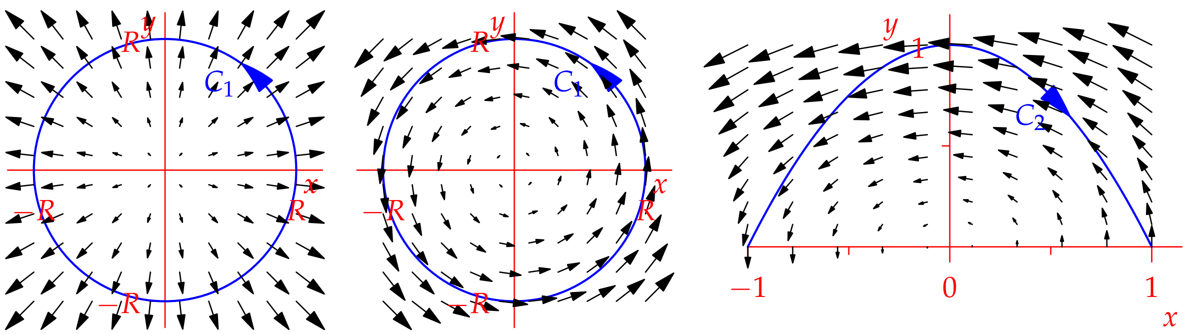
(a) The work done by the radial field $\mathbf{F}_a(x,y) = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ on the particle is

$$W_a = \int_0^{2\pi} R^2(-\sin t\cos t + \cos t\sin t)\,\mathrm{d}t = 0$$

(b) By contrast, the work done by the rotational field $\mathbf{F}_b(x,y) = \left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)$ is

$$W_b = \int_0^{2\pi} R^2(\sin t^2 + \cos^2 t)\,\mathrm{d}t = 2\pi R^2$$

Both particles make the same journey: in the first case the travel is perpendicular to the field, so $\mathbf{F}_a$ neither helps nor hinders the journey; in the second case, the field $\mathbf{F}_b$ provides a tailwind.



1 (a): $W_a = 0$ 1 (b): $W_b > 0$ 2: $W_2 < 0$

2. A particle moves along the parabola C_2 with equation $y = 1 - x^2$ between $(-1,0)$ and $(1,0)$. Parametrizing this by $\mathbf{r}(t) = \left(\begin{smallmatrix} t \\ 1-t^2 \end{smallmatrix}\right)$ with $-1 \le t \le 1$, we see that the work done by the same rotational field $\mathbf{F}_b(x,y) = \left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)$ in moving a particle along C_2 is

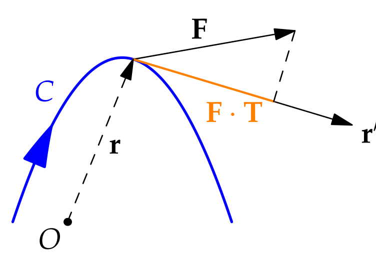
$$W_2 = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_C \begin{pmatrix} -y \\ x \end{pmatrix}\cdot\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \end{pmatrix} = \int_{-1}^{1} \begin{pmatrix} t^2-1 \\ t \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -2t \end{pmatrix}\mathrm{d}t = \int_{-1}^{1} -t^2 - 1\,\mathrm{d}t = -\frac{8}{3}$$

The work done is negative since the particle generally moves against the push of the field.

Physical Significance: Why "Work"?

Work done is *energy* transferred. A force field $\mathbf{F}$ accelerates a particle, causing it to gain or lose kinetic energy (motion) in accordance with Newton's second law ($\mathbf{F} = m\mathbf{r}''$).

The contribution of $\mathbf{F}$ to accelerating the particle along C is the component that points in the direction of travel: the dot product $\mathbf{F}\cdot\frac{\mathbf{r}'}{|\mathbf{r}'|} = \mathbf{F}\cdot\mathbf{T}$, where $\mathbf{T}$ is the unit tangent vector.



The kinetic energy gained by the particle over an infinitesimal arc-length $\mathrm{d}s$ is then

$$\mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s = \mathbf{F}\cdot\frac{\mathbf{r}'}{|\mathbf{r}'|}\,|\mathbf{r}'|\,\mathrm{d}t = \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

The work done by the field is the net gain of kinetic energy, the sum of all infinitesimal contributions:

$$W = \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s = \int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

Orientation and Parameterization

To evaluate every integral thus far, we first made an explicit *choice of parametrization* $\mathbf{r}(t)$. But this merely measures *how fast* and in what *direction* we travel along the curve (its *orientation*). What happens if we change these things?

Theorem. *Suppose* $\int_C f\,\mathrm{d}s$ *or* $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ *are line integrals along a curve C.*

1. *The values of both integrals are independent of* ***orientation-preserving*** *re-parametrizations of C.*
2. *Label by $-C$ the same curve but with its* ***orientation reversed****. Then*

$$\int_{-C} f\,\mathrm{d}s = \int_C f\,\mathrm{d}s \quad \textit{and} \quad \int_{-C} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = -\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

The upshot is that you can use any orientation-preserving parametrization you like and nothing will change! As a sanity check, think about how *arc-length* doesn't care in which direction you measure a curve, but that the *work done* switches sign if you push against the wind…

Proof. Suppose $\mathbf{r}(t)$ parametrizes C with $a \le t \le b$. Any other parametrization has the form

$$\mathbf{R}(T) = \mathbf{r}\big(t(T)\big)$$

where $t(T)$ is a 1–1, onto function. There are two cases.

1. Orientation-preserving: $t(T)$ is increasing, $\frac{\mathrm{d}t}{\mathrm{d}T} > 0$, and $t^{-1}(a) \le T \le t^{-1}(b)$. Now compute:

$$\begin{aligned}
\int_{t^{-1}(a)}^{t^{-1}(b)} f\big(\mathbf{R}(T)\big)\left|\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}T}\right|\mathrm{d}T &= \int_{t^{-1}(a)}^{t^{-1}(b)} f\big(\mathbf{r}(t(T))\big)\left|\frac{\mathrm{d}}{\mathrm{d}T}\mathbf{r}\big(t(T)\big)\right|\mathrm{d}T \\
&= \int_{t^{-1}(a)}^{t^{-1}(b)} f\big(\mathbf{r}(t(T))\big)\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}T}\right|\mathrm{d}T && \text{(chain rule)} \\
&= \int_{t^{-1}(a)}^{t^{-1}(b)} f\big(\mathbf{r}(t(T))\big)\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|\frac{\mathrm{d}t}{\mathrm{d}T}\,\mathrm{d}T && (\text{since } \tfrac{\mathrm{d}t}{\mathrm{d}T} > 0) \\
&= \int_a^b f\big(\mathbf{r}(t)\big)\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right|\mathrm{d}t && \text{(substitution rule)}
\end{aligned}$$

   The values of $\int_C f\,\mathrm{d}s$ are the same regardless of the parametrization.

2. Orientation-reversing: $t(T)$ is decreasing, $\frac{\mathrm{d}t}{\mathrm{d}T} < 0$, and $t^{-1}(b) \le T \le t^{-1}(a)$. The calculation is identical, except that the limits on the integral are reversed and that $\left|\frac{\mathrm{d}t}{\mathrm{d}T}\right| = -\frac{\mathrm{d}t}{\mathrm{d}T}$. Both changes contribute a negative sign, whence the total value of the integral is unchanged.

For the force integral, write $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\,\mathrm{d}s$ and observe that $\mathbf{T}$ changes sign when the orientation is reversed.

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