

16.2 Line Integrals

Definition. Let f be a continuous scalar function. Let C be an oriented curve in 2-space parameterized by

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad a \leq t \leq b$$

where $x(t), y(t)$ have continuous first derivatives. The line integral of f along C is

$$\int_C f \, ds := \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2} \, dt$$

The definition is identical in 3-dimensions, f is now a function of three variables:

$$\int_C f \, ds := \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt$$

Notation and Arc-length The letter s usually indicates *distance* or *displacement* in Physics. Suppose that a particle has traveled a distance $s(t)$ along C at time t . Since the rate of change of distance is speed, we conclude that

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

Integrating both sides with respect to t must recover the arc-length of the curve

$$s = \int_a^b \frac{ds}{dt} \, dt = \int_a^b |\mathbf{r}'(t)| \, dt$$

Therefore, when $f = 1$ the integral along C simply returns the length of the curve C

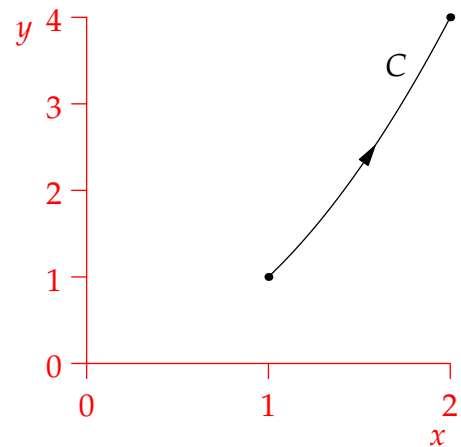
Example Find $\int_C \frac{y}{x} \, ds$, where C is the parabola $y = x^2$ for $1 \leq x \leq 2$

First parameterize the curve: $\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, then compute

$$ds = \sqrt{x'(t)^2 + y'(t)^2} \, dt = \sqrt{1 + 4t^2} \, dt$$

Then

$$\begin{aligned} \int_C \frac{y}{x} \, ds &= \int_1^2 \frac{t^2}{t} \cdot \sqrt{1 + 4t^2} \, dt = \int_1^2 t(1 + 4t^2)^{1/2} \, dt \\ &= \frac{1}{3/2} \cdot \frac{1}{8} [1 + 4t^2]^{3/2} \Big|_1^2 = \frac{1}{12} (17^{3/2} - 5^{3/2}) \end{aligned}$$



Example A length of wire is wound twice round into a helix C of radius R with a distance $2\pi h$ between spirals. It is parameterized by

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} R \cos t \\ R \sin t \\ ht \end{pmatrix}$$

where $0 \leq t \leq 4\pi$. If the density of the wire is ρ then its mass is

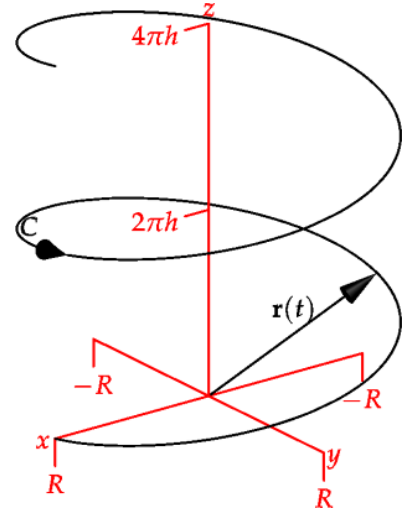
$$\begin{aligned} \int_C \rho(\mathbf{r}(t)) \, ds &= \int_0^{4\pi} \rho(\mathbf{r}(t)) \sqrt{R^2 \sin^2 t + R^2 \cos^2 t + h^2} \, dt \\ &= \sqrt{R^2 + h^2} \int_0^{4\pi} \rho(\mathbf{r}(t)) \, dt \end{aligned}$$

Suppose that ρ is constant, then the mass is simply

$$\int_C \rho(\mathbf{r}(t)) \, ds = \sqrt{R^2 + h^2} \int_0^{4\pi} \rho(\mathbf{r}(t)) \, dt = 4\pi\rho\sqrt{R^2 + h^2}$$

If instead the density is $\rho(x, y, z) = z(4\pi h - z)$ (thinner at the ends of the wire), then the mass is

$$\begin{aligned} \int_C \rho(\mathbf{r}(t)) \, ds &= \sqrt{R^2 + h^2} \int_0^{4\pi} z(4\pi h - z) \, dt = \sqrt{R^2 + h^2} \int_0^{4\pi} ht(4\pi h - ht) \, dt \\ &= h^2 \sqrt{R^2 + h^2} \int_0^{4\pi} 4\pi t - t^2 \, dt = h^2 \sqrt{R^2 + h^2} \cdot \frac{1}{6} (4\pi)^3 = \frac{32\pi^3}{3} h^2 \sqrt{R^2 + h^2} \end{aligned}$$



Interpretation: Riemann Sums and Averages Fix $n \in \mathbb{N}$ and suppose that $(x_0, y_0), \dots, (x_n, y_n)$ are a sequence of equally-spaced points along the curve C and define

$$\Delta s_i = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

to be the distance between successive points. Note that $\Delta s_1 = \Delta s_2 = \Delta s_3 = \dots = \Delta s_n$. The Riemann sum definition of the line integral of f along C is then

$$\begin{aligned} \int_C f \, ds &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i, y_i) \Delta s_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{f(x_i, y_i)}{n} n \Delta s_i \\ &= \left(\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{f(x_i, y_i)}{n} \right) \left(\lim_{n \rightarrow \infty} n \Delta s_1 \right) = \left(\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{f(x_i, y_i)}{n} \right) (\text{arc-length of } C) \end{aligned}$$

The first term is simply the average value of the function f along the curve C .

We can think of the above example as computing the average density of the wire multiplied by its length.

Example A hiker walks over a high mountain pass modelled by the equation $y = 4 - \frac{x^2}{9}$ from $x = -3$ to $x = 3$, with distance measured in km. The air pressure at a height y km above sea level is modelled by

$$P = 100 \left(1 - \frac{y}{50} \right)^{5.3} \text{ kPa}$$

Find an expression for the average air pressure along the hiking trail.

The trail may be parameterized by $\mathbf{r}(t) = \left(\frac{t}{4 - \frac{t^2}{9}} \right)$ for $-3 \leq t \leq 3$. Thus the average air pressure is

$$P_{\text{av}} = \frac{\int_{-3}^3 P(\mathbf{r}(t)) |\mathbf{r}'(t)| dt}{\int_{-3}^3 |\mathbf{r}'(t)| dt} = \frac{1}{\int_{-3}^3 \sqrt{1 + \frac{4}{81}t^2} dt} \int_{-3}^3 100 \left(1 - \frac{4 - \frac{t^2}{9}}{50} \right)^{5.3} \sqrt{1 + \frac{4}{81}t^2} dt$$

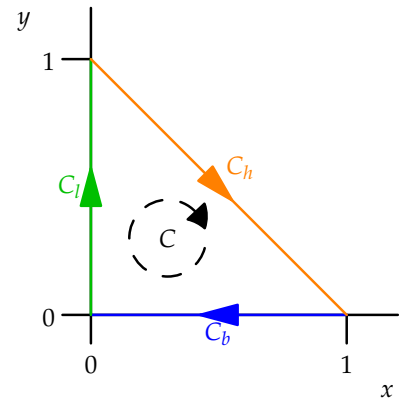
The numerator cannot be computed exactly, but with the assistance of a computer we find that

$$P_{\text{av}} = 66.95 \text{ kPa, to 2 d.p.}$$

One might interpret this as saying that the hiker will have, on average, $\frac{2}{3}$ of the oxygen they would have available at sea-level, or that climbing the pass would be 50% harder than an equivalent climb at sea-level.

Piecewise smooth curves If C is piecewise smooth split the integral into pieces

Example Find the line integral of $f(x, y) = x^2(y + 1)$ over the triangle C shown



We must parameterize each piece of the curve separately.

Base: $\mathbf{r}_b(t) = (1-t, 0)$, $0 \leq t \leq 1$ gives

$$\int_{C_b} f ds = \int_0^1 (1-t)^2 \sqrt{(-1)^2 + 0^2} dt = \frac{1}{3}$$

Left: $f = 0 \implies \int_{C_l} f ds = 0$

Hypotenuse: $\mathbf{r}_h(t) = (1-t, t)$, $0 \leq t \leq 1$ gives

$$\int_{C_h} f ds = \int_0^1 t^2(1-t+1) \sqrt{1^2 + 1^2} dt = \sqrt{2} \int_0^1 2t^2 - t^3 dt = \frac{5\sqrt{2}}{12}$$

The total integral is therefore $\int_C x^2(y + 1) ds = \frac{1}{3} + \frac{5\sqrt{2}}{12}$

Direction and Parameterization In the previous example, we could have computed the line integral in the opposite direction and we'd have obtained *exactly the same answer!* Indeed, it should also be noticed that the choice of parameterization of a curve doesn't matter. You should always obtain the same value for a line integral $\int_C f ds$.

Theorem. The value of a line integral $\int_C f ds$ is independent of parameterization and of orientation (direction) of the curve C .

Proof. Suppose $\mathbf{r}(t)$ parameterizes C with $a \leq t \leq b$. Then any other parameterization of the curve will take the form

$$\mathbf{R}(T) = \mathbf{r}(t(T))$$

where $t(T)$ is a 1-1, onto function. There are two cases.

1. $t(T)$ is increasing. This corresponds to parameterizing the curve with the same orientation as with $\mathbf{r}(t)$. We have $\frac{dt}{dT} > 0$, and $\mathbf{R}(T) = \mathbf{r}(t(T))$ with $t^{-1}(a) \leq T \leq t^{-1}(b)$.

But then

$$\begin{aligned} \int_C f \, ds &= \int_{t^{-1}(a)}^{t^{-1}(b)} f(\mathbf{R}(T)) |\mathbf{R}'(T)| \, dT = \int_{t^{-1}(a)}^{t^{-1}(b)} f(\mathbf{r}(t(T))) \left| \frac{d}{dT} \mathbf{r}(t(T)) \right| \, dT \\ &= \int_{t^{-1}(a)}^{t^{-1}(b)} f(\mathbf{r}(t(T))) \left| \frac{d\mathbf{r}}{dt} \cdot \frac{dt}{dT} \right| \, dT && \text{(chain rule)} \\ &= \int_{t^{-1}(a)}^{t^{-1}(b)} f(\mathbf{r}(t(T))) \left| \frac{d\mathbf{r}}{dt} \right| \frac{dt}{dT} \, dT && \text{(since } \frac{dt}{dT} > 0) \\ &= \int_a^b f(\mathbf{r}(t)) \left| \frac{d\mathbf{r}}{dt} \right| \, dt && \text{(substitution rule)} \end{aligned}$$

This is precisely the definition of $\int_C f \, ds$ parameterized using $\mathbf{r}(t)$.

2. $t(T)$ is decreasing. This corresponds to parameterizing the curve with the reverse orientation to $\mathbf{r}(t)$. We have $\frac{dt}{dT} < 0$, and $\mathbf{R}(T) = \mathbf{r}(t(T))$ with $t^{-1}(b) \leq T \leq t^{-1}(a)$.

The calculation is identical, except that the limits on the integral are reversed, and that $\left| \frac{dt}{dT} \right| = -\frac{dt}{dT}$. Both these changes contribute a negative sign, whence the total value of the integral is unchanged. ■

Line Integrals with respect to co-ordinate functions There is a second type of line integral.

Definition. The line integrals along C of f with respect to x, y, z respectively are the integrals

$$\begin{aligned} \int_C f \, dx &:= \int_a^b f(x(t), y(t), z(t)) x'(t) \, dt \\ \int_C f \, dy &:= \int_a^b f(x(t), y(t), z(t)) y'(t) \, dt \\ \int_C f \, dz &:= \int_a^b f(x(t), y(t), z(t)) z'(t) \, dt \end{aligned}$$

Warning! In contrast to line integrals with respect to arc-length, these new integrals do depend on the orientation of the curve C : if you reverse the direction then these new integrals *change sign*.

Interpretation As long as the x co-ordinate of a point on C is always moving in the same direction, we may interpret

$$\int_C f \, dx = (\text{average value of } f \text{ along } C) \cdot (\text{change in } x \text{ along } C)$$

However, if the curve C starts to double back on itself, this interpretation doesn't hold.

Work Line integrals with respect to x, y, z most often appear in Physics.

Definition. The work done by a vector field \mathbf{F} in moving a particle along a path C is the integral

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$\int_C \mathbf{F} \cdot d\mathbf{r}$ also known as the line integral of \mathbf{F} along C

Under our notation $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ we compute $d\mathbf{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$. Suppose that $\mathbf{F} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$. Then

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \int_C P dx + Q dy + R dz$$

A work integral is therefore the sum of three¹ integrals with respect to co-ordinates.

It is often easier to leave the integral in vector form when substituting in the parameterization. Once everything is written in terms of t , multiply out the dot product. For example, if C is parameterized by $\mathbf{r}(t)$ with $a \leq t \leq b$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \begin{pmatrix} P(\mathbf{r}(t)) \\ Q(\mathbf{r}(t)) \\ R(\mathbf{r}(t)) \end{pmatrix} \cdot \frac{d\mathbf{r}}{dt} dt$$

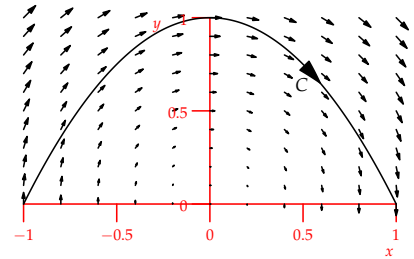
Example Find the work done by the vector field $\mathbf{F}(x, y) = \begin{pmatrix} y \\ -x \end{pmatrix}$ in moving a particle along the parabola

$$y = 1 - x^2 \quad \text{for} \quad -1 \leq x \leq 1$$

Let $\mathbf{r}(t) = \begin{pmatrix} t \\ 1-t^2 \end{pmatrix}$, so that $d\mathbf{r} = \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 1 \\ -2t \end{pmatrix} dt$.

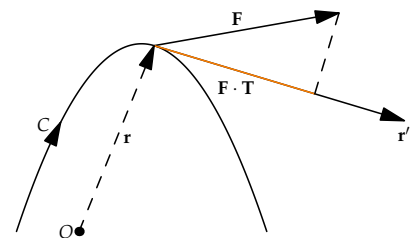
The work done is then

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \begin{pmatrix} y \\ -x \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} = \int_{-1}^1 \begin{pmatrix} 1-t^2 \\ -t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2t \end{pmatrix} dt = \int_{-1}^1 1 - t^2 + 2t^2 dt = \frac{8}{3}$$



Physical Significance of Work A force field \mathbf{F} causes a moving particle to gain/lose kinetic energy (motion)

If a particle has velocity vector \mathbf{r}' at position \mathbf{r} , the contribution of the force field \mathbf{F} to accelerating the particle is the dot product $\mathbf{F} \cdot \frac{\mathbf{r}'}{|\mathbf{r}'|} = \mathbf{F} \cdot \mathbf{T}$, where \mathbf{T} is the unit tangent vector to the curve.



The kinetic energy gained by the particle over an infinitesimal arc-length ds is then

$$\mathbf{F} \cdot \mathbf{T} ds = \mathbf{F} \cdot \frac{\mathbf{r}'}{|\mathbf{r}'|} |\mathbf{r}'| dt = \mathbf{F} \cdot d\mathbf{r}$$

The work done is then the sum of all these infinitesimal contributions

$$W = \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r}$$

¹Or two in 2D.

Examples

1. Find the work done by the vector field $\mathbf{F}(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$ around one orbit of the circle $x^2 + y^2 = a^2$

The path is $\mathbf{r}(t) = \begin{pmatrix} a \cos t \\ a \sin t \end{pmatrix}$ and so $d\mathbf{r} = \begin{pmatrix} -a \sin t \\ a \cos t \end{pmatrix} dt$.
The work done is therefore

$$W = \int_0^{2\pi} a^2 (-\sin t \cos t + \cos t \sin t) dt = 0$$

2. Now let $\mathbf{F}(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix}$ along the same path

The work is now

$$W = \int_0^{2\pi} a^2 (\sin^2 t + \cos^2 t) dt = 2\pi a^2$$

In the first example the force points at right-angles to the direction of travel, so the force doesn't help the particle. In the second example, the force is always pushing the particle in the direction of travel, so it does work on the particle's behalf.

