16.3 The Fundamental Theorem of Line Integrals

The Fundamental Theorem of Calculus says that we may evaluate the integral of a derivative simply by knowing the values of the function at the *endpoints* of the interval of integration:

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$

The Fundamental Theorem of Line Integrals is an analogue of this for multi-variable functions.

Theorem (Fundamental Theorem of Line Integrals)**.** *Suppose that C is a smooth curve from points A to B parametrized by $\mathbf{r}(t)$ for $a \le t \le b$. Let f be a differentiable function whose domain includes C and whose gradient vector ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(B) - f(A)$$

Alternatively: *if $\mathbf{F}$ is a continuous conservative vector field with potential function f then*

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(\textit{end of } C) - f(\textit{start of } C)$$

A line integral in a conservative vector field is *independent of path*: its value depends only on the endpoints of the curve, **not** on the path between them. This idea is really important and we'll return to it shortly.

The long caveats about differentiability and continuity in the Theorem's statement are merely so that the original Fundamental Theorem of Calculus can be invoked in the proof.

Proof. ($n = 2$ or 3 for the purposes of this course)

$$\begin{aligned}
\int_C \nabla f \cdot \mathrm{d}\mathbf{r} &= \int_C \begin{pmatrix} f_{x_1} \\ \vdots \\ f_{x_n} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x_1 \\ \vdots \\ \mathrm{d}x_n \end{pmatrix} = \int_C \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n}\mathrm{d}x_n \\
&= \int_a^b \left(\frac{\partial f}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t} \right) \mathrm{d}t \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(x_1(t), \ldots, x_n(t))\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t))\mathrm{d}t \qquad \text{(chain rule)} \\
&= f\big(\mathbf{r}(b)\big) - f\big(\mathbf{r}(a)\big)
\end{aligned}$$

where we applied FTC in the final step.

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Examples 1. Let C be the curve parametrized by $\mathbf{r}(t) = \left(\begin{smallmatrix} 1+\sin^2 t \\ t+\sin t \end{smallmatrix}\right)$ for $0 \le t \le 2\pi$. Then

$$\int_C \nabla(x^2y^3) \cdot \mathrm{d}\mathbf{r} = x^2y^3\Big|_{(1,0)}^{(1,2\pi)} = 8\pi^3 - 0 = 8\pi^3$$

2. Let C be parametrized by $\mathbf{r}(t) = \begin{pmatrix} t^3-1 \\ t-t^{-1} \end{pmatrix}$ for $2 \le t \le 3$. Then

$$\int_C \sin y \, dx + x\cos y \, dy = \int_C \nabla(x\sin y)\cdot d\mathbf{r} = x\sin y\Big|_{(7,\frac{3}{2})}^{(26,\frac{8}{3})} = 26\sin\frac{8}{3} - 7\sin\frac{3}{2}$$

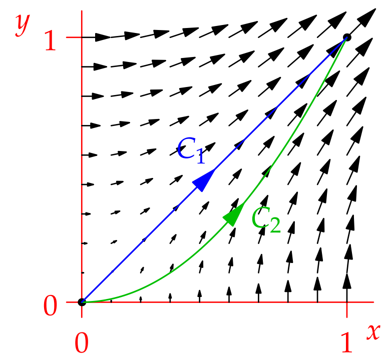
3. Evaluate the line integrals $\int_{C_i} y\,dx + x\,dy$ where C_1 is the straight line from $(0,0)$ and $(1,1)$, and C_2 is the parabola $y = x^2$ between the same points.

   We may parametrize the first curve $\mathbf{r}(t) = \begin{pmatrix} t \\ t \end{pmatrix}$, so

$$\int_{C_1} y\,dx + x\,dy = \int_0^1 2t\,dt = 1$$

   For the second $\mathbf{r}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, so

$$\int_{C_2} y\,dx + x\,dy = \int_0^1 t^2\,dt + 2t^2\,dt = 1$$



   We expected the two solutions to be the same since $\begin{pmatrix} y \\ x \end{pmatrix} = \nabla(xy)$ is conservative. We could instead simply have applied the Fundamental Theorem:

$$\int_{C_i} \begin{pmatrix} y \\ x \end{pmatrix}\cdot d\mathbf{r} = \int_{C_i} \nabla(xy)\cdot d\mathbf{r} = xy\Big|_{(0,0)}^{(1,1)} = 1$$

Conservation of Energy

The terminology we use (conservative, potential, etc.) all comes from Physics.
Suppose that a particle of mass m follows a path C through a conservative force field $\mathbf{F} = -\nabla f$.
Parametrize C so that the particle is at time t its position is $\mathbf{r}(t)$ and its *velocity* is $\mathbf{v}(t) = \mathbf{r}'(t)$.
The particle has *kinetic energy* $K = \frac{1}{2}m|\mathbf{v}|^2$ and is said to have *potential energy* f.
We evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ in two ways:

1. Newton's second law ($\mathbf{F} = m\mathbf{a} = m\mathbf{v}'$) says that[1]

$$\int_C \mathbf{F}\cdot d\mathbf{r} = m\int_{t_0}^{t_1} \mathbf{v}'(t)\cdot\mathbf{v}(t)\,dt = m\int_{t_0}^{t_1} \frac{d}{dt}\frac{1}{2}|\mathbf{v}|^2\,dt = \frac{1}{2}m|\mathbf{v}|^2\Big|_{t_0}^{t_1} = \triangle K$$

   is the change in kinetic energy over the path.

2. By the Fundamental Theorem,

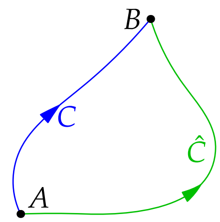
$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_C \nabla f\cdot d\mathbf{r} = -f\big(\mathbf{r}(t)\big)\Big|_{t_0}^{t_1} = -\triangle f$$

   is negative the change in potential energy of the particle over the path.

We conclude that $\triangle f + \triangle K = 0$: **total energy is conserved**. This is why potential functions in Physics tend to have a *negative sign:* $\mathbf{F} = -\nabla f$ (in mathematics, we omit the negative).

[1]By the product rule, $\frac{d}{dt}\mathbf{v}\cdot\mathbf{v} = \mathbf{v}'\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}' = 2\mathbf{v}'\cdot\mathbf{v}$.

Path Independence



The Fundamental Theorem has an amazing interpretation: line integrals in conservative vector fields depend only on a curve's *endpoints*. We'd like to turn this idea on its head. Does a line integral (or integrals?) being independent of path require a vector field to be conservative?

This discussion is lengthy and tricky, but very important.

Definition. A line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is *independent of path* if $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\hat{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ for *all* curves $\hat{C}$ with the same endpoints as C.

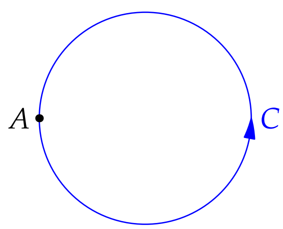
Before stating the relevant theorem, we need a few terms form basic topology.

Definition. 1. A curve is *closed* if it starts and finishes at the same point.

2. Let D be a region: for us, this means a subset of either the plane or 3D space. We say that D is:

   (a) *Open* if it contains none of its boundary points.

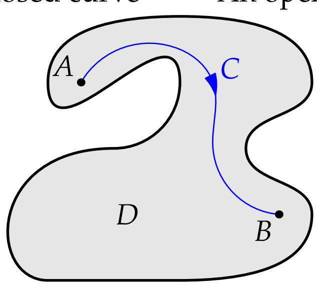
   (b) *(Path-)Connected* if every pair of points A, B in D can be joined by a curve lying entirely in D.



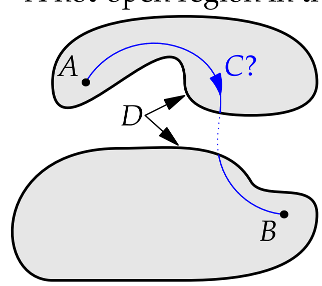
A closed curve

An open region in the plane

A not-open region in the plane



A connected region: C joins points A, B



A disconnected region: we cannot join A, B with a curve lying entirely in D

For instance, the inside of the unit disk $D = \{(x, y) : x^2 + y^2 < 1\}$ is both connected and open. Its boundary curve (the circle $x^2 + y^2 = 1$) is closed.

Theorem. *Let C be a curve in an open connected region D and* $\mathbf{F}$ *a continuous vector field on D. The following statements are equivalent* (*mean the same thing*):

1. $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ *is independent of path.*

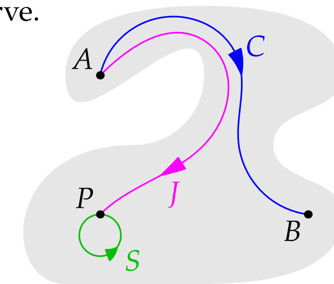
2. $\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ *for* every closed *curve S in D.*

3. $\mathbf{F}$ *is conservative.*

Read the statement carefully: if even one line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path, then *all line integrals* over *all curves* in D are independent of path. Path-independence has doesn't depend on paths at all: it is a property of the *vector field*.

Proof. We prove the theorem in three stages.

(1 ⇒ 2) Suppose $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path and let S be a closed curve. Since D is connected, we may join the starting point A of C to some fixed point P on S by a curve J lying in D. Writing $-J$ for the same curve travelled in reverse, we see that the composite curve



$$C_1 = J \cup S \cup (-J) \cup C$$

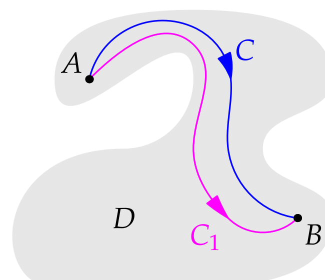
has the same endpoints (A, B) as C. By path-independence,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \left[\int_J + \int_S + \int_{-J} + \int_C\right]\mathbf{F}\cdot d\mathbf{r} = \int_S \mathbf{F}\cdot d\mathbf{r} + \int_C \mathbf{F}\cdot d\mathbf{r}$$

since $\int_{-J} \mathbf{F}\cdot d\mathbf{r} = -\int_J \mathbf{F}\cdot d\mathbf{r}$. We conclude that $\int_S \mathbf{F}\cdot d\mathbf{r} = 0$.

(2 ⇒ 3) Assume $\int_S \mathbf{F}\cdot d\mathbf{r} = 0$ around every closed curve in D. Choose any point A in D and define

$$f(x, y) = \int_C \mathbf{F}\cdot d\mathbf{r}$$

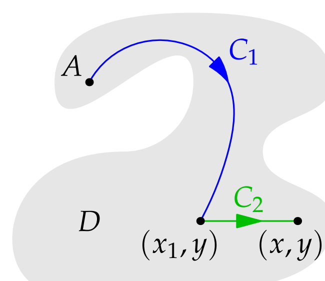


where C is any curve joining A to (x, y). If $\tilde{C}$ is another curve with the same endpoints, then $S = C \cup (-\tilde{C})$ is a closed curve, whence

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \left(\int_C - \int_S\right)\mathbf{F}\cdot d\mathbf{r} = \int_{\tilde{C}} \mathbf{F}\cdot d\mathbf{r}$$

The function f is therefore independent[2] of the choice of path C. We claim that f is a potential function for $\mathbf{F}$.

Since D is open, there exists $(x_1, y) \in D$ such that $x_1 < x$. Let C_1 join A to (x_1, y), and C_2 be the line segment thence to (x, y). Then



$$f(x, y) = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

Since x_1 is a constant, the first integral is independent of x and so

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

Write $\mathbf{F} = \binom{P}{Q}$. Along C_2 we have y constant, hence $dy = 0$. Therefore

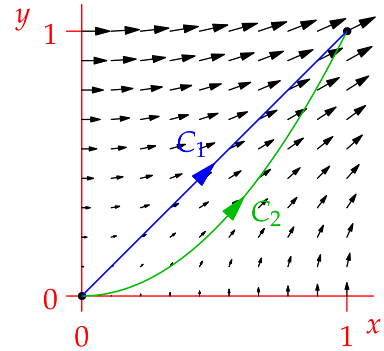
$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int_{(x_1,y)}^{(x,y)} P\,dx + Q\,dy = \frac{\partial}{\partial x}\int_{(x_1,y)}^{(x,y)} P\,dx = \frac{d}{dx}\int_{x_1}^{x} P(t, y)\,dt = P(x, y)$$

by the Fundamental Theorem of Calculus. A similar argument shows that $Q(x, y) = \frac{\partial f}{\partial y}$ (in 3D a z-argument is also needed). Putting this together we see that $\mathbf{F} = \nabla f$ is conservative.

(3 ⇒ 1) This is simply the Fundamental Theorem of Line Integrals. ∎

[2] This (and the first part) show that (1 ⇔ 2) even without the assumptions about D being open and $\mathbf{F}$ continuous.

Example We evaluate the line integrals $\int_{C_i} \begin{pmatrix} 2y \\ x \end{pmatrix} \cdot \mathrm{d}\mathbf{r}$ over the line $(0,0) \to (1,1)$ and the parabola $y = x^2$ between the same two points.



For the first curve, $\mathbf{r}(t) = \begin{pmatrix} t \\ t \end{pmatrix}$ produces

$$\int_{C_1} 2y\,\mathrm{d}x + x\,\mathrm{d}y = \int_0^1 3t\,\mathrm{d}t = \frac{3}{2}$$

For the second curve, $\mathbf{r}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$ yields

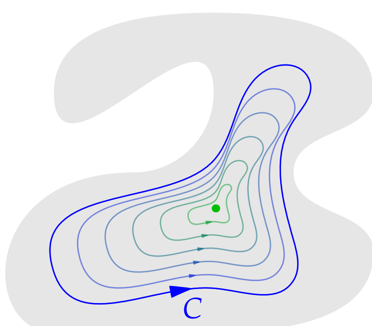
$$\int_{C_2} 2y\,\mathrm{d}x + x\,\mathrm{d}y = \int_0^1 2t^2 + 2t^2\,\mathrm{d}t = \frac{4}{3} \neq \frac{3}{2}$$

The relative strengths of the arrows along each curve should make it clear why $\int_{C_2} < \int_{C_1}$. The fact that these integrals have different values says that $\mathbf{F} = \begin{pmatrix} 2y \\ x \end{pmatrix}$ is *not* conservative.

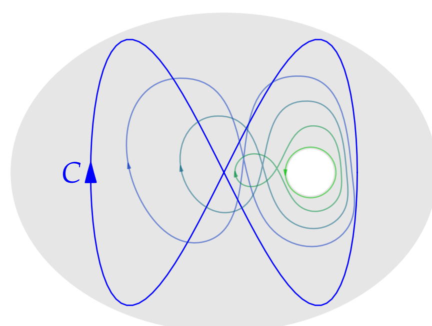
Simple-Connectedness: When is F conservative?

Searching for a potential function can be a lot of work. It is useful to know if a vector field is conservative *before* you start looking. This discussion requires one more topological concept.

Definition. A connected region D is *simply-connected* if every closed curve in D may be shrunk to a point *without leaving* D.



Simply-connected: any closed curve may be shrunk to a point



Not simply-connected: cannot shrink C to a point due to the 'hole' in the region

Theorem. *Suppose* $\mathbf{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$ *has continuous partial derivatives on a domain* D.

1. *If* $\mathbf{F}$ *is conservative, then* $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.
2. *If* D *is simply-connected and* $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, *then* $\mathbf{F}$ *is conservative.*

Proof. 1. Suppose $\mathbf{F} = \nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$. Then $Q_x = f_{yx} = f_{xy} = P_y$.

2. This will follow quickly from Green's Theorem (next section).

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Examples 1. $\mathbf{F} = \begin{pmatrix} 2y \\ x \end{pmatrix}$ is not conservative (on any domain) because $\frac{\partial}{\partial x} x = 1 \neq 2 = \frac{\partial}{\partial y}(2y)$.

2. Find a potential function, if there is one, for the vector field on $\mathbf{F} = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}$ on $D = \mathbb{R}^2$.

   You can dive straight in to the computation, but it is helpful to use the Theorem first so that you don't waste time. Since $\mathbb{R}^2$ is simply-connected,

   $$\frac{\partial Q}{\partial x} = 2x = \frac{\partial P}{\partial y} \implies \mathbf{F} \text{ is conservative}$$

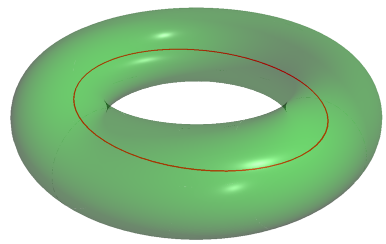
   Now that we know a potential function f exists, we can solve for it:

   $$f_x = P = 2xy \implies f(x,y) = x^2y + g(y)$$
   $$f_y = Q = x^2 \implies f(x,y) = x^2y + h(x)$$

   for unknown functions g, h. Choosing $g(y) = h(x) = c$ (constant) yields all possible potential functions $f(x,y) = x^2y + c$.

   From this, it follows that $\int_C 2xy\,\mathrm{d}x + x^2\,\mathrm{d}y = 0$ around any closed curve S.

What changes in 3D? Essentially nothing!



Simple-connectedness means the same thing. For example, a solid torus is non-simply-connected since a curve can be drawn inside it which cannot be shrunk to a point.

The primary difference is that the relevant Theorem has three conditions, corresponding to the three pairs of mixed partial derivatives…

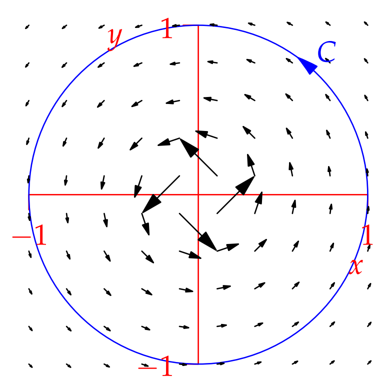
Theorem. *Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on a simply-connected region $E \subseteq \mathbb{R}^3$, and that P, Q, R have continuous partial derivatives. Then $\mathbf{F}$ is conservative if and only if*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \quad \textit{and} \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$

This isn't so useful to us and will be much easier to remember once we've introduced curl later on…

Does Simple-Connectedness Really Matter? Yes!

To see why, consider applying our analysis to the vector field $\mathbf{F} = \begin{pmatrix} P \\ Q \end{pmatrix} = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$.



1. Calculate the line integral of $\mathbf{F}$ around the unit circle:

   $\mathbf{r}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$, $0 \le t \le 2\pi$

   $$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \mathrm{d}t = 2\pi$$

   Since $\int_C \mathbf{F} \cdot \mathrm{d}r \neq 0$, we believe that $\mathbf{F}$ is non-conservative.

2. However, the partial derivatives are equal

   $$\frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial P}{\partial y}$$

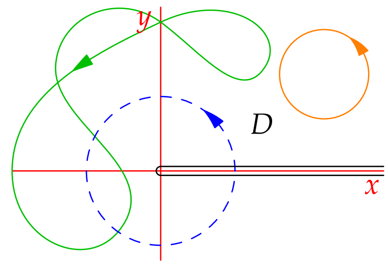
   We therefore want to say that $\mathbf{F}$ is conservative.

One of these arguments must be false, but which? The answer depends on whether our *domain* is *simply-connected.* A full argument requires Green's Theorem, but we can see the shape of it already…

Case 1: D **encircles the origin** Suppose that $D = \mathbb{R}^2 \setminus \{(0,0)\}$ is the punctured plane (without the origin). Such a domain D is *not simply-connected*. Indeed the unit circle cannot be shrunk to a point. The $Q_x = P_y$ doesn't allow us to conclude anything. Argument 1 is correct and the vector field is non-conservative on D.

Case 2: D **is simply-connected** For instance, we could exclude the positive x-axis and let

$$D = \mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$$



Now argument 2 is correct: $\mathbf{F}$ is conservative and $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every closed curve in D.
In the picture, line integrals round the solid curves are zero; the unit circle does not lie in D. A suitable potential function for $\mathbf{F}$ on D is $f = \theta$ (the polar-angle):

$$f(x,y) = \begin{cases} \tan^{-1} \frac{y}{x} & y > 0 \\ \pi & y = 0 \\ \pi + \tan^{-1} \frac{y}{x} & y < 0 \end{cases}$$

The problem in extending this potential function to the entire punctured plane is that θ is not continuous everywhere. Indeed it is typically taken to be discontinuous on the positive x-axis!

Summary

If $\mathbf{F}(x,y) = P\mathbf{i} + Q\mathbf{j}$ is defined on an open connected region $D \subseteq \mathbb{R}^2$, where P, Q have continuous first derivatives, then the following are equivalent:

1. $\mathbf{F}$ is conservative ($= \nabla f$ for some potential function f).
2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path for any curve C in D.
3. $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise smooth closed curve C in D.
4. In addition, if D is simply-connected, then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ on a volume $E \subseteq \mathbb{R}^3$ then the fifth condition becomes

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$