16.3 The Fundamental Theorem of Line Integrals

Recall the Fundamental Theorem of Calculus for a single-variable function $f$:

$$\int_{a}^{b} f'(x) \, dx = f(b) - f(a)$$

It says that we may evaluate the integral of a derivative simply by knowing the values of the function at the endpoints of the interval of integration $[a, b]$. The Fundamental Theorem of Line Integrals is a precise analogue of this for multi-variable functions. The primary change is that gradient $\nabla f$ takes the place of the derivative $f'$ in the original theorem.

**Theorem** (Fundamental Theorem of Line Integrals). Suppose that $C$ is a smooth curve from points $A$ to $B$ parameterized by $\mathbf{r}(t)$ for $a \leq t \leq b$. Let $f$ be a differentiable function whose domain includes $C$ and whose gradient vector $\nabla f$ is continuous on $C$. Then

$$\int_{C} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(B) - f(A)$$

The long caveats about differentiability and continuity are merely so that the original Fundamental Theorem of Calculus can be invoked in the proof.

**Proof.** ($n = 2$ or $3$ for the purposes of this course)

$$\int_{C} \nabla f \cdot d\mathbf{r} = \int_{C} \left( \frac{\partial f}{\partial x_1} \, dx_1 + \cdots + \frac{\partial f}{\partial x_n} \, dx_n \right) = \int_{C} \frac{\partial f}{\partial x_1} \, dx_1 + \cdots + \frac{\partial f}{\partial x_n} \, dx_n$$

$$= \int_{a}^{b} \left( \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt} \right) \, dt$$

$$= \int_{a}^{b} \frac{df}{dt}(x_1(t), \ldots, x_n(t)) \, dt = \int_{a}^{b} \frac{df(\mathbf{r}(t))}{dt} \, dt \quad \text{(chain rule)}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

where we applied FTC in the final step.

The Theorem can be alternatively stated: if $\mathbf{F}$ is a conservative vector field with potential function $f$ then

$$\int_{C} \mathbf{F} \cdot d\mathbf{r} = f(\text{end of } C) - f(\text{start of } C)$$

We say that a line integral in a conservative vector field is independent of path.

**Examples**

1. Let $C$ be the curve parameterized by $\mathbf{r}(t) = \left( \frac{1+\sin^2 t}{1+\sin t} \right)$ for $0 \leq t \leq 2\pi$. Then

$$\int_{C} \nabla(x^2y^3) \cdot d\mathbf{r} = x^2y^3 \bigg|_{(1,0)} = 8\pi^3 - 0 = 8\pi^3$$
2. Let \( C \) be the curve parameterized by \( \mathbf{r}(t) = \left( \frac{t^3-1}{t-1}, \frac{t^3}{t-1} \right) \) for \( 2 \leq t \leq 3 \). Then

\[
\int_C \sin y \, dx + x \cos y \, dy = \int_C \nabla(x \sin y) \cdot d\mathbf{r} = x \sin y \bigg|_{(26, \frac{3}{2})}^{(26, \frac{3}{2})} = 26 \sin \frac{3}{2} - 7 \sin \frac{3}{2}
\]

3. Evaluate the line integrals \( \int_C y \, dx + x \, dy \) where \( C_1 \) is the straight line from \((0,0)\) and \((1,1)\), and \( C_2 \) is the parabola \( y = x^2 \) between the same points.

For the first curve we have \( \mathbf{r}(t) = (t, t) \), so

\[
\int_{C_1} y \, dx + x \, dy = \int_0^1 2t \, dt = 1
\]

For the second curve we have \( \mathbf{r}(t) = (t, t^2) \), so

\[
\int_{C_2} y \, dx + x \, dy = \int_0^1 t^2 \, dt + 2t^2 \, dt = 1
\]

We expected the two solutions to be the same since \( (\frac{x}{y}) = \nabla(xy) \) is conservative. We could simply have applied the Fundamental Theorem:

\[
\int_{C_1} \left( \begin{array}{c} y \\ x \end{array} \right) \cdot d\mathbf{r} = \int_{C_1} \nabla(xy) \cdot d\mathbf{r} = xy \bigg|_{(0,0)}^{(1,1)} = 1 - 0 = 1
\]

4. Evaluate \( \int_C y^2 z \, dx + 2xyz \, dy + xy^2 \, dz \) along any curve joining \((1,0,0)\) and \((2,1,-1)\).

The integral is \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} = \nabla(xy^2z) \), so the path is irrelevant and we obtain

\[
\int_C y^2 z \, dx + 2xyz \, dy + xy^2 \, dz = xy^2 z \bigg|_{(1,0,0)}^{(2,1,-1)} = -2
\]

**Conservation of Energy** The terminology (conservative, potential, etc.) all comes from Physics.

There are two primary forms of energy: potential (stored) and kinetic (motion).

Suppose that a particle of mass \( m \) follows a curve \( C \) through a conservative force field \( \mathbf{F} = -\nabla f \).

We parameterize the curve so that the particle is at position \( \mathbf{r}(t) \) at time \( t \). Its velocity vector is then \( \mathbf{v}(t) = \frac{d\mathbf{r}}{dt} \).

The particle has kinetic energy \( K = \frac{1}{2} m |\mathbf{v}|^2 \) and is said to have potential energy \( f \).

Now we evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) in two ways.

1. Newton’s second law \( (\mathbf{F} = m\mathbf{a} = mv') \) says that\(^1\)

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = m \int_{t_0}^{t_1} \mathbf{v}'(t) \cdot \mathbf{v}(t) \, dt = m \int_{t_0}^{t_1} \frac{d}{dt} \frac{1}{2} |\mathbf{v}|^2 \, dt = \frac{1}{2} m |\mathbf{v}|^2 \bigg|_{t_0}^{t_1} = \Delta K
\]

is the change in kinetic energy over the path.

\(^1\)By the product rule, \( \frac{d}{dt} \mathbf{v} \cdot \mathbf{v} = \mathbf{v}' \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}' = 2\mathbf{v}' \cdot \mathbf{v} = 2|\mathbf{v}|^2 \).
2. Alternatively we may use the Fundamental Theorem:

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = - \int_C \nabla f \cdot d\mathbf{r} = -f(r(t)) \bigg|_{t_0}^{t_1} = - \Delta f \]

is negative the change in potential energy of the particle over the path.

Therefore \( \Delta f + \Delta K = 0 \), and so total energy is conserved. Since Physicists always want energy to be conserved, they typically choose potential functions to have a negative sign: \( \mathbf{F} = -\nabla f \). In mathematics, we omit the negative.

Path Independence

The Fundamental Theorem has the amazing interpretation that line integrals in conservative vector fields depend only on a curve’s endpoints. We want to turn this idea on its head. Is it the case that a line integral (or integrals?) being independent of path forces a vector field to be conservative?

**Definition.** A line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path if \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\hat{C}} \mathbf{F} \cdot d\mathbf{r} \) for any curve \( \hat{C} \) with the same endpoints as \( C \).

Before we can state the relevant theorems, we need to understand the meaning of several terms.

**Definition.** A region \( D \) is open if it contains no boundary points.

For example, the inside of the unit disk \( D = \{(x, y) : x^2 + y^2 < 1\} \) is open.

**Definition.** A region \( D \) is (path-)connected if every pair of points \( A, B \) in \( D \) can be joined by a curve lying entirely in \( D \).

A connected region with curve \( C \) joining \( A, B \)   A disconnected region: cannot join \( A, B \) with a curve lying in \( D \).
The adjectives *open* and *connected* apply only to *domains/regions* in this course. The final adjective applies only to *curves*.

**Definition.** A curve is closed if it starts and finishes at the same point.

**Independence of Path and Closed Curves** The following important Theorem relates being independent of path to line integrals round closed curves.

**Theorem.** Let $C$ be a curve in a connected region $D$. Then $\int_C F \cdot dr$ is independent of path if and only if $\int_S F \cdot dr = 0$ for every closed path $S$ in $D$.

Read the Theorem carefully: if even one curve $C$ has $\int_C F \cdot dr$ independent of path, then the line integrals over *all curves* must be independent of path. This says that independence of path is really a property of the *vector field* $F$ rather than a specific curve.

**Proof.** Suppose first that $\int_C F \cdot dr$ is independent of path and let $S$ be a closed curve in $D$. Since $D$ is connected, we may join the starting point $A$ of $C$ to some fixed point $P$ on $S$ by a curve $J$ lying in $D$. Writing $-J$ for the same curve travelled in reverse, we see that the composite curve $C_1 = J \cup S \cup (-J) \cup C$ has the same endpoints as $C$. By independence of path, it follows that

$$\int_C F \cdot dr = \int_{C_1} F \cdot dr = \left[ \int_J + \int_S + \int_{-J} + \int_C \right] F \cdot dr$$

However $\int_{-J} F \cdot dr = -\int_J F \cdot dr$, since travelling in reverse changes the sign of a work integral. Cancelling these and the $\int_C F \cdot dr$ terms from both sides forces us to conclude that $\int_S F \cdot dr = 0$.

Conversely, suppose that the line integral round any closed curve $S$ evaluates to zero. Let $C$ and $C_1$ be two curves with the same endpoints. Then $C \cup (-C_1)$ is a closed curve $S$, whence

$$\int_C F \cdot dr - \int_{C_1} F \cdot dr = 0.$$

Thus $\int_C F \cdot dr$ is independent of path, regardless of $C$. □
Finally, we see that independence of path is equivalent to the vector field being conservative.

**Theorem.** Let \( \mathbf{F} \) be a continuous vector field on, and \( C \) a curve in, an open, connected region. Then \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path if and only if \( \mathbf{F} \) is conservative.

**Proof of Theorem (for 2 dimensions).** If \( \mathbf{F} \) is conservative with potential function \( f \) then the Fundamental Theorem tells us that \( \int_C \mathbf{F} \cdot d\mathbf{r} \) depends only on the endpoints of \( C \).

Conversely, suppose that \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path. By the previous Theorem, \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path for any curve \( C \) in \( D \). Choose a point \( A \) and define \( f(x,y) = \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is any curve joining \( A \) to \((x,y)\). The function \( f \) is well-defined because \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path. We claim that \( f \) is a potential function for \( \mathbf{F} \).

Choose \((x,y) \in D \). Since \( D \) is open, there exists a point \((x_1,y) \in D \) such that \( x_1 < x \). Let \( C_1 \) be a path from \( A \) to \((x_1,y) \) and \( C_2 \) the line segment thence to \((x,y) \). Then

\[
f(x,y) = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}
\]

Since \( x_1 \) is constant, the first integral is independent of \( x \) and so

\[
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot d\mathbf{r}
\]

Now let \( \mathbf{F} = \left( \frac{P}{Q} \right) \). Along the curve \( C_2 \) we have \( y \) constant, hence \( dy = 0 \). Therefore

\[
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{(x_1,y)}^{(x,y)} P \, dx + Q \, dy = \frac{\partial}{\partial x} \int_{(x_1,y)}^{(x,y)} P \, dx = \frac{d}{dx} \int_{x_1}^{x} P(t,y) \, dt = P(x,y)
\]

by the Fundamental Theorem of Calculus.

A similar argument (choose \((x,y_1) \in D \) with \( y_1 < y \)) shows that \( Q(x,y) = \frac{\partial f}{\partial y} \). Putting this together we see that \( \nabla f = \mathbf{F} \) and so \( \mathbf{F} \) is conservative.

The proof in three dimensions requires a similar third argument for \( \frac{\partial f}{\partial z} \).

**Example** Evaluate the line integrals \( \int_{C_1} \left( \frac{2y}{x} \right) \cdot d\mathbf{r} \) over the same line and parabola as before.

For the first curve we have \( \mathbf{r}(t) = (t, t^2) \), whence

\[
\int_{C_1} 2y \, dx + x \, dy = \int_0^1 3t \, dt = \frac{3}{2}
\]

For the second curve we have \( \mathbf{r}(t) = (t^2, t^2) \), and so

\[
\int_{C_2} 2y \, dx + x \, dy = \int_0^1 2t^2 \, dt + \int_0^1 2t^2 \, dt = \frac{4}{3} \neq \frac{3}{2}
\]

Considering the strength of the arrows in the picture it should be clear why \( \int_{C_2} < \int_{C_1} \). The fact that these integrals give different values tells us that \( \mathbf{F} = \left( \frac{2y}{x} \right) \) is not conservative.
When is $F$ conservative? Searching for a potential function can involve a lot of work. It is useful to know if a vector field is conservative before you start looking.

**Theorem.** Suppose that $P, Q$ have continuous first derivatives on a region $D$, then

$$F = \begin{pmatrix} P \\ Q \end{pmatrix} \text{ conservative } \implies \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \text{ throughout } D.$$ 

Equivalently, $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$ $\implies$ $F = \begin{pmatrix} P \\ Q \end{pmatrix}$ not conservative.

**Proof.** Suppose $F = \nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$. Then $f_{xy} = f_{yx}$, hence result.

While the Theorem can easily tell us that a vector field is not conservative, it doesn’t quite work in reverse. We first need some more topology.

**Simply-connected regions**

**Definition.** A connected region $D$ is simply-connected if and only if every closed curve in $D$ may be shrunk to a point without leaving $D$.

**Theorem** (Proof requires Green’s Theorem). Suppose $F = \begin{pmatrix} P \\ Q \end{pmatrix}$ has continuous partial derivatives on a simply-connected region $D$ and suppose that $P, Q$ have continuous partial derivatives. Then $F$ is conservative if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ throughout $D$.

**Examples**

1. Find a potential function, if there is one, for the vector field on $F = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}$ on $D = \mathbb{R}^2$.

You can dive straight in to computing, but it is helpful to use the Theorem first so that you don’t waste time.

$$\frac{\partial Q}{\partial x} = 2x = \frac{\partial P}{\partial y} \implies F \text{ is conservative.}$$

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Now that we know a potential function \( f \) exists, we can solve for it:

\[
\begin{align*}
    f_x &= P = 2xy \implies f(x, y) = x^2y + g(y) \\
    f_y &= Q = x^2 \implies f(x, y) = x^2y + h(x)
\end{align*}
\]

for unknown functions \( g, h \). Choosing \( g(y) = h(x) = c \) (constant) yields all possible potential functions \( f(x, y) = x^2y + c \).

2. \( \mathbf{F}(x, y) = \left( \frac{y \sin x}{x \sin x} \right) \) has domain \( D = \mathbb{R}^2 \). We quickly see that

\[
\begin{align*}
    \frac{\partial Q}{\partial x} &= \sin x + x \cos x \quad \text{and} \quad \frac{\partial P}{\partial y} = \sin x
\end{align*}
\]

\[
\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y} \implies \mathbf{F} \text{ is non-conservative.}
\]

3. Let \( \mathbf{F}(x, y) = \left( \frac{2x + x^2y}{x^2e^{xy}} \right) \). Prove that \( \int_C \mathbf{F} \cdot \text{d}r = 0 \) where \( C \) is any closed curve in the plane

We calculate the mixed partial derivatives:

\[
\begin{align*}
    \frac{\partial}{\partial x} x^3e^{xy} &= 3x^2e^{xy} + x^3ye^{xy} = x^2e^{xy}(3 + xy) \\
    \frac{\partial}{\partial y} (2x + x^2y)e^{xy} &= x^2e^{xy} + (2x + x^2y)xe^{xy} = x^2e^{xy}(3 + xy)
\end{align*}
\]

These are equal, hence \( \mathbf{F} \) is conservative and consequently all line integrals over closed paths \( C \) evaluate to zero.

What changes in three dimensions? Essentially nothing! A connected volume \( E \) is still simply-connected if every curve can be shrunk to a point without leaving \( E \).

The interior of a solid torus \( E \) is non-simply-connected since a curve can be drawn inside it which cannot be shrunk to a point.

The primary difference is that the corresponding Theorem is not quite as easy to use as the 2D version...

**Theorem.** Suppose \( \mathbf{F} = Pi + Qj + Rk \) is a vector field on a simply-connected region \( E \subseteq \mathbb{R}^3 \) and suppose that \( P, Q, R \) have continuous partial derivatives. Then \( \mathbf{F} \) is conservative iff

\[
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \text{and} \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \text{throughout } E.
\]

The proof requires Stokes’ Theorem and will be much easier to remember once we’ve introduced Curl. At the present it is not worth remembering.
Example  Let \( F(x, y) = \left( \frac{2x+2y+6xyz+z}{2x+3x^2z \cdot 3x^2y^2+3x^2y} \right) \). Writing \( F = Pi + Qj + Rk \) we can laboriously check that \( P_y = Q_x \), etc., to see that \( F \) is indeed conservative. Writing \( F = \nabla f \) we can partially integrate:

\[
\begin{align*}
  f_x &= P = 2x + 2y + 6xyz + z \\
  \implies f(x, y, z) &= x^2 + 2xy + 3x^2yz + xz + g(y, z) \\
  f_y &= Q = 2x + 3x^2z \\
  \implies f(x, y, z) &= 2xy + 3x^2yz + h(x, z) \\
  f_z &= R = 3x^2y + x \\
  \implies f(x, y, z) &= 3x^2yz + xz + j(x, y)
\end{align*}
\]

for unknown functions \( g, h, j \). Suitable choices yield

\[
f(x, y, z) = x^2 + 2xy + 3x^2yz + xz
\]

Does Simple-Connectedness Really Matter? Let us analyze the vector field \( F = \frac{1}{x^2 + y^2} \left( -\frac{y}{x} \right) \) and try to decide if it is conservative.

1. First compute the partial derivatives:

\[
\begin{align*}
  \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x} \frac{x}{x^2 + y^2} = \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \\
  \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y} \frac{-y}{x^2 + y^2} = \frac{-1}{x^2 + y^2} + \frac{2y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}
\end{align*}
\]

Since \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \) we want to say that \( F \) is conservative.

2. Now calculate the line integral of \( F \) around the unit circle: \( \mathbf{r}(t) = \left( \frac{\cos t}{\sin t} \right), \ 0 \leq t \leq 2\pi \)

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \left( -\sin t \right) \left( -\sin t \right) \ d t = 2\pi
\]

Since \( \int_C \mathbf{F} \cdot d\mathbf{r} \neq 0 \) we conclude that \( F \) is non-conservative.

At least one of these arguments has to be false, but which? The answer, strangely, depends on your choice of domain.

- With no additional information, you should assume that the domain of the vector field is the largest set possible, in this case the punctured plane \( D = \mathbb{R}^2 \setminus \{(0,0)\} \), the plane with the
origin removed. This is non-simply-connected; indeed the unit circle cannot be shrunk to a
point without some part of it passing through the origin! The easy $Q_x = P_y$ theorem does not
apply and argument 1. is false. The vector field is conservative.

- Suppose that the domain was restricted so that it was simply
connected. For instance, we could exclude the positive $x$-
axis and choose the domain

$$D = \mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$$

Now $D$ is simply-connected and $\mathbf{F}$ is conservative. Moreover
$f_C \mathbf{F} \cdot \mathbf{dr} = 0$ for every closed curve in $D$. Note that the original unit circle is no longer a curve in $D$, whence argument
2. is now false.

In the picture, the line integrals round the solid curves are zero, while around the blue curve
(which is not in $D$) the line integral evaluates to $2\pi$. You can easily check that a suitable potential function for $\mathbf{F}$ on $D$ is $f = \theta$ in polar-coordinates; otherwise said,

$$f(x,y) = \begin{cases} \tan^{-1} \frac{y}{x} & y > 0 \\ \pi & y = 0 \\ \pi + \tan^{-1} \frac{y}{x} & y < 0 \end{cases}$$

The problem in extending this potential function to the entire punctured plane is that $\theta$ is not
continuous everywhere; typically it is discontinuous on the positive $x$-axis.

**Summary** If $\mathbf{F}(x,y) = Pi + Qj$ is defined on a connected region $D \subseteq \mathbb{R}^2$, where $P, Q$ have continuous first derivatives then the following are equivalent:

1. $\mathbf{F}$ is conservative
2. $\mathbf{F}$ is a gradient field ($= \nabla f$ for some potential function $f$)
3. $\int_C \mathbf{F} \cdot \mathbf{dr}$ is independent of path for any curve $C$ in $D$
4. $\int_C \mathbf{F} \cdot \mathbf{dr} = 0$ for every piecewise smooth closed curve $C$ in $D$

In addition, if $D$ is simply-connected we also have

5. $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

If $\mathbf{F} = Pi + Qj + Rk$ on a volume $E \subseteq \mathbb{R}^3$ then the fifth condition becomes

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$