16.3 The Fundamental Theorem of Line Integrals

Recall the Fundamental Theorem of Calculus for a single-variable function f:

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$

It says that we may evaluate the integral of a derivative simply by knowing the values of the function at the endpoints of the interval of integration $[a, b]$.
The Fundamental Theorem of Line Integrals is a precise analogue of this for multi-variable functions. The primary change is that *gradient* ∇f takes the place of the derivative f' in the original theorem.

Theorem (Fundamental Theorem of Line Integrals)**.** *Suppose that C is a smooth curve from points A to B parameterized by $\mathbf{r}(t)$ for $a \le t \le b$. Let f be a differentiable function whose domain includes C and whose gradient vector ∇f is continuous on C. Then*

$$\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(B) - f(A)$$

The long caveats about differentiability and continuity are merely so that the original Fundamental Theorem of Calculus can be invoked in the proof.

Proof. ($n = 2$ or 3 for the purposes of this course)

$$\begin{aligned}\int_C \nabla f \cdot \mathrm{d}\mathbf{r} &= \int_C \begin{pmatrix} f_{x_1} \\ \vdots \\ f_{x_n} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}x_1 \\ \vdots \\ \mathrm{d}x_n \end{pmatrix} = \int_C \frac{\partial f}{\partial x_1}\,\mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n}\,\mathrm{d}x_n \\ &= \int_a^b \left(\frac{\partial f}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t} \right) \mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(x_1(t), \ldots, x_n(t))\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}(t))\,\mathrm{d}t \qquad \text{(chain rule)} \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a))\end{aligned}$$

where we applied FTC in the final step. ■

The Theorem can be alternatively stated: if $\mathbf{F}$ is a conservative vector field with potential function f then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(\text{end of } C) - f(\text{start of } C)$$

We say that a line integral in a conservative vector field is *independent of path.*

Examples

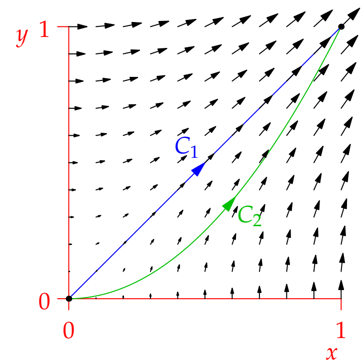
1. Let C be the curve parameterized by $\mathbf{r}(t) = \left(\begin{smallmatrix} 1+\sin^2 t \\ t+\sin t \end{smallmatrix}\right)$ for $0 \le t \le 2\pi$. Then

$$\int_C \nabla(x^2y^3) \cdot \mathrm{d}\mathbf{r} = x^2y^3\Big|_{(1,0)}^{(1,2\pi)} = 8\pi^3 - 0 = 8\pi^3$$

2. Let C be the curve parameterized by $\mathbf{r}(t) = \begin{pmatrix} t^3-1 \\ t-t^{-1} \end{pmatrix}$ for $2 \le t \le 3$. Then

$$\int_C \sin y \, \mathrm{d}x + x \cos y \, \mathrm{d}y = \int_C \nabla(x \sin y) \cdot \mathrm{d}\mathbf{r} = x \sin y \Big|_{(7,\frac{3}{2})}^{(26,\frac{8}{3})} = 26 \sin \frac{8}{3} - 7 \sin \frac{3}{2}$$

3. Evaluate the line integrals $\int_{C_i} y \, \mathrm{d}x + x \, \mathrm{d}y$ where C_1 is the straight line from (0,0) and (1,1), and C_2 is the parabola $y = x^2$ between the same points.



For the first curve we have $\mathbf{r}(t) = \binom{t}{t}$, so

$$\int_{C_1} y \, \mathrm{d}x + x \, \mathrm{d}y = \int_0^1 2t \, \mathrm{d}t = 1$$

For the second curve we have $\mathbf{r}(t) = \binom{t}{t^2}$, so

$$\int_{C_2} y \, \mathrm{d}x + x \, \mathrm{d}y = \int_0^1 t^2 \, \mathrm{d}t + 2t^2 \, \mathrm{d}t = 1$$

We expected the two solutions to be the same since $\binom{y}{x} = \nabla(xy)$ is conservative. We could simply have applied the Fundamental Theorem:

$$\int_{C_i} \begin{pmatrix} y \\ x \end{pmatrix} \cdot \mathrm{d}\mathbf{r} = \int_{C_i} \nabla(xy) \cdot \mathrm{d}\mathbf{r} = xy \Big|_{(0,0)}^{(1,1)} = 1 - 0 = 1$$

4. Evaluate $\int_C y^2 z \, \mathrm{d}x + 2xyz \, \mathrm{d}y + xy^2 \, \mathrm{d}z$ along any curve joining $(1, 0, 0)$ and $(2, 1, -1)$.

The integral is $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where $\mathbf{F} = \nabla(xy^2 z)$, so the path is irrelevant and we obtain

$$\int_C y^2 z \, \mathrm{d}x + 2xyz \, \mathrm{d}y + xy^2 \, \mathrm{d}z = xy^2 z \Big|_{(1,0,0)}^{(2,1,-1)} = -2$$

Conservation of Energy The terminology (conservative, potential, etc.) all comes from Physics.

There are two primary forms of energy: potential (stored) and kinetic (motion).
Suppose that a particle of mass m follows a curve C through a conservative force field $\mathbf{F} = -\nabla f$.
We parameterize the curve so that the particle is at position $\mathbf{r}(t)$ at time t. Its *velocity vector* is then

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

The particle has *kinetic energy* $K = \frac{1}{2} m |\mathbf{v}|^2$ and is said to have *potential energy* f.

Now we evaluate the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ in two ways.

1. Newton's second law ($\mathbf{F} = m\mathbf{a} = m\mathbf{v}'$) says that[1]

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = m \int_{t_0}^{t_1} \mathbf{v}'(t) \cdot \mathbf{v}(t) \, \mathrm{d}t = m \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} |\mathbf{v}|^2 \, \mathrm{d}t = \frac{1}{2} m |\mathbf{v}|^2 \Big|_{t_0}^{t_1} = \triangle K$$

is the change in kinetic energy over the path.

[1] By the product rule, $\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{v} \cdot \mathbf{v} = \mathbf{v}' \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}' = 2\mathbf{v}' \cdot \mathbf{v} = 2 |\mathbf{v}|^2$.

2. Alternatively we may use the Fundamental Theorem:

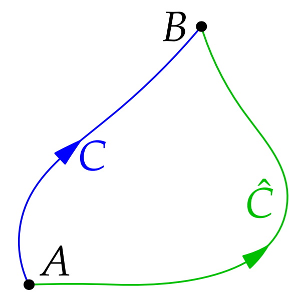
$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C \nabla f \cdot d\mathbf{r} = -f(\mathbf{r}(t))\Big|_{t_0}^{t_1} = -\triangle f$$

is negative the change in potential energy of the particle over the path.

Therefore $\triangle f + \triangle K = 0$, and so total energy is conserved. Since Physicists always want energy to be conserved, they typically choose potential functions to have a *negative sign:* $\mathbf{F} = -\nabla f$. In mathematics, we omit the negative.

Path Independence

The Fundamental Theorem has the amazing interpretation that line integrals in conservative vector fields depend only on a curve's endpoints. We want to turn this idea on its head. Is it the case that a line integral (or integrals?) being independent of path forces a vector field to be conservative?



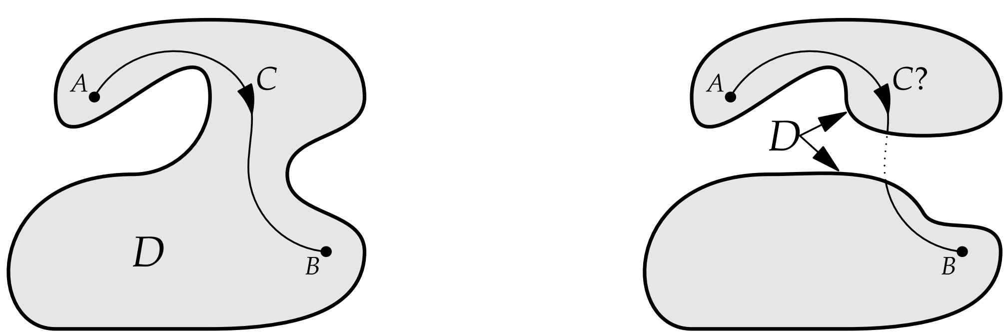
Definition. *A line integral* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *is* independent of path *if* $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\hat{C}} \mathbf{F} \cdot d\mathbf{r}$ *for* any *curve* $\hat{C}$ *with the same endpoints as* C

Before we can state the relevant theorems, we need to understand the meaning of several terms.

Definition. *A region D is* open *if it contains no boundary points.*

For example, the inside of the unit disk $D = \{(x, y) : x^2 + y^2 < 1\}$ is open.

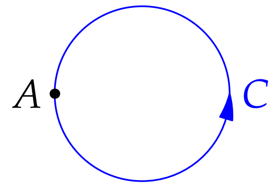
Definition. *A region D is* (path-)connected *if every pair of points A, B in D can be joined by a curve lying entirely in D.*



A connected region with curve C joining A, B

A disconnected region: cannot join A, B with a curve lying in D.

The adjectives *open* and *connected* apply only to *domains/regions* in this course. The final adjective applies only to *curves.*



Definition. *A curve is* closed *if it starts and finishes at the same point.*

Independence of Path and Closed Curves The following important Theorem relates being independent of path to line integrals round closed curves.

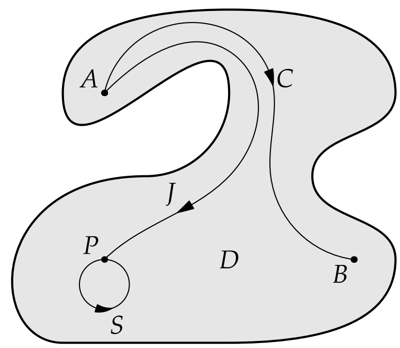
Theorem. *Let C be a curve in a connected region D. Then $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path if and only if $\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for every* closed *path S in D.*

Read the Theorem carefully: if even one curve C has $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ independent of path, then the line integrals over *all curves* must be independent of path. This says that independence of path is really a property of the *vector field* $\mathbf{F}$ rather than a specific curve.

Proof. Suppose first that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path and let S be a closed curve in D. Since D is connected, we may join the starting point A of C to some fixed point P on S by a curve J lying in D. Writing $-J$ for the same curve travelled in reverse, we see that the composite curve

$$C_1 = J \cup S \cup (-J) \cup C$$

has the same endpoints as C. By independence of path, it follows that



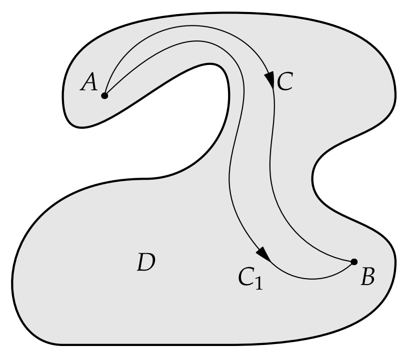
$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \left[\int_J + \int_S + \int_{-J} + \int_C \right] \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

However $\int_{-J} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_J \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, since travelling in reverse changes the sign of a work integral. Cancelling these and the $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ terms from both sides forces us to conclude that $\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$.

Conversely, suppose that the line integral round any closed curve S evaluates to zero. Let C and C_1 be two curves with the same endpoints. Then $C \cup (-C_1)$ is a closed curve S, whence

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0.$$

Thus $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path, regardless of C. ■



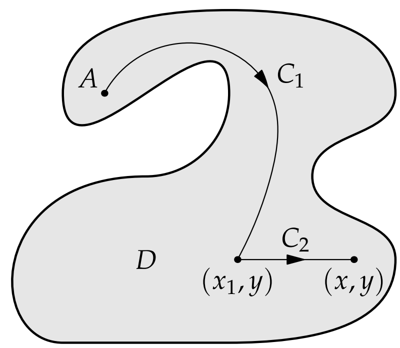
Finally, we see that independence of path is equivalent to the vector field being conservative.

Theorem. *Let* $\mathbf{F}$ *be a continuous vector field on, and* C *a curve in, an open, connected region. Then* $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ *is independent of path if and only if* $\mathbf{F}$ *is conservative.*

Proof of Theorem (for 2 dimensions). If $\mathbf{F}$ is conservative with potential function f then the Fundamental Theorem tells us that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ depends only on the endpoints of C.

Conversely, suppose that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path. By the previous Theorem, $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path for *any* curve C in D. Choose a point A and define $f(x,y) = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is any curve joining A to (x,y). The fucntion f is well-defined because $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of path. We claim that f is a potential function for $\mathbf{F}$.

Choose $(x,y) \in D$. Since D is open, there exists a point $(x_1,y) \in D$ such that $x_1 < x$. Let C_1 be a path from A to (x_1,y) and C_2 the line segment thence to (x,y). Then



$$f(x,y) = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Since x_1 is constant, the first integral is independent of x and so

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

Now let $\mathbf{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$. Along the curve C_2 we have y constant, hence $\mathrm{d}y = 0$. Therefore

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{(x_1,y)}^{(x,y)} P\,\mathrm{d}x + Q\,\mathrm{d}y = \frac{\partial}{\partial x} \int_{(x_1,y)}^{(x,y)} P\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}x} \int_{x_1}^{x} P(t,y)\,\mathrm{d}t = P(x,y)$$

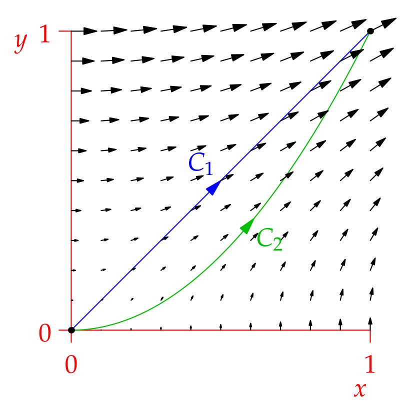
by the Fundamental Theorem of Calculus.

A similar argument (choose $(x,y_1) \in D$ with $y_1 < y$) shows that $Q(x,y) = \frac{\partial f}{\partial y}$. Putting this together we see that $\nabla f = \mathbf{F}$ and so $\mathbf{F}$ is conservative.

■

The proof in three dimensions requires a similar third argument for $\frac{\partial f}{\partial z}$.

Example Evaluate the line integrals $\int_{C_i} \begin{pmatrix} 2y \\ x \end{pmatrix} \cdot \mathrm{d}\mathbf{r}$ over the same line and parabola as before.



For the first curve we have $\mathbf{r}(t) = \begin{pmatrix} t \\ t \end{pmatrix}$, whence

$$\int_{C_1} 2y\,\mathrm{d}x + x\,\mathrm{d}y = \int_0^1 3t\,\mathrm{d}t = \frac{3}{2}$$

For the second curve we have $\mathbf{r}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$, and so

$$\int_{C_2} 2y\,\mathrm{d}x + x\,\mathrm{d}y = \int_0^1 2t^2\,\mathrm{d}t + \int_0^1 2t^2\,\mathrm{d}t = \frac{4}{3} \neq \frac{3}{2}$$

Considering the strength of the arrows in the picture it should be clear why $\int_{C_2} < \int_{C_1}$. The fact that these integrals give different values tells us that $\mathbf{F} = \begin{pmatrix} 2y \\ x \end{pmatrix}$ is *not* conservative.

When is F conservative? Serching for a potential function can involve a lot of work. It is useful to know if a vector field is conservative *before* you start looking.

Theorem. *Suppose that P, Q have continuous first derivatives on a region D, then*

$$\mathbf{F} = \begin{pmatrix} P \\ Q \end{pmatrix} \text{ conservative } \implies \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \text{ throughout } D.$$

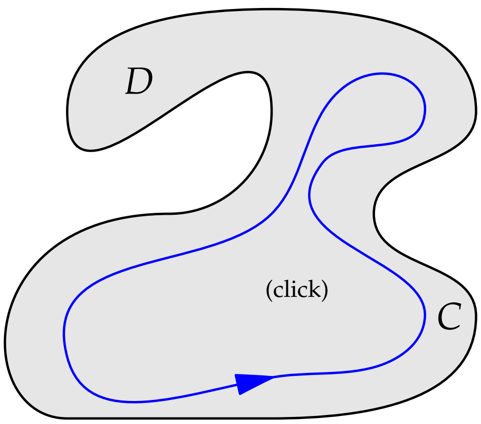
Equivalently: $\dfrac{\partial Q}{\partial x} \neq \dfrac{\partial P}{\partial y} \implies \mathbf{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$ not *conservative.*

Proof. Suppose $\mathbf{F} = \nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$. Then $f_{xy} = f_{yx}$, hence result. ∎

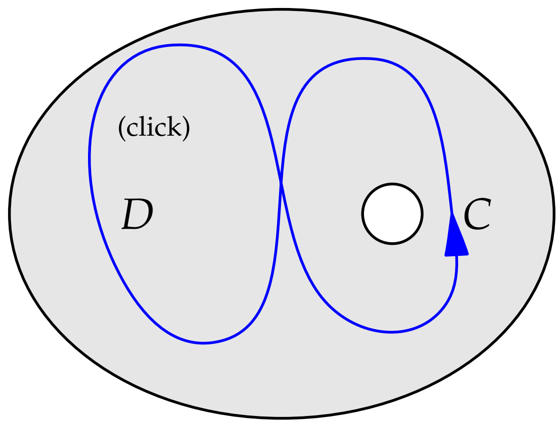
While the Theorem can easily tell us that a vector field is *not* conservative, it doesn't quite work in reverse. We first need some more topology.

Simply-connected regions

Definition. *A connected region D is* simply-connected *if and only if every closed curve in D may be shrunk to a point* without leaving *D.*



D simply-connected



D connected but not simply-connected: cannot shrink C to a point due to the 'hole' in D

Theorem (Proof requires Green's Theorem)**.** *Suppose* $\mathbf{F} = \begin{pmatrix} P \\ Q \end{pmatrix}$ *has continuous partial derivatives on a simply-connected region D and suppose that P, Q have continuous partial derivatives. Then* $\mathbf{F}$ *is conservative if and only if* $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ *throughout D.*

Examples

1. Find a potential function, if there is one, for the vector field on $\mathbf{F} = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}$ on $D = \mathbb{R}^2$.

   You can dive straight in to computing, but it is helpful to use the Theorem first so that you don't waste time.

   $$\frac{\partial Q}{\partial x} = 2x = \frac{\partial P}{\partial y} \implies \mathbf{F} \text{ is conservative.}$$

Now that we know a potential function f exists, we can solve for it:

$$f_x = P = 2xy \implies f(x,y) = x^2y + g(y)$$
$$f_y = Q = x^2 \implies f(x,y) = x^2y + h(x)$$

for unknown functions g, h. Choosing $g(y) = h(x) = c$ (constant) yields all possible potential functions $f(x,y) = x^2y + c$.

2. $\mathbf{F}(x,y) = \begin{pmatrix} y\sin x \\ x\sin x \end{pmatrix}$ has domain $D = \mathbb{R}^2$. We quickly see that

$$\frac{\partial Q}{\partial x} = \sin x + x\cos x \quad \text{and} \quad \frac{\partial P}{\partial y} = \sin x$$

$\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y} \implies$ **F** is non-conservative.

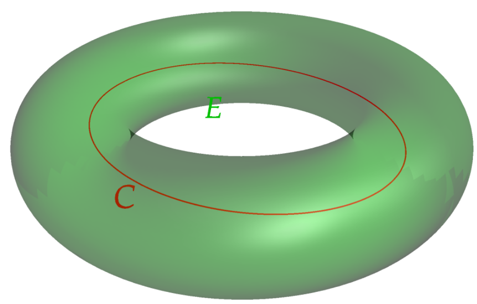
3. Let $\mathbf{F}(x,y) = \begin{pmatrix} (2x+x^2y)e^{xy} \\ x^3e^{xy} \end{pmatrix}$. Prove that $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ where C is any closed curve in the plane

We calculate the mixed partial derivatives:

$$\frac{\partial}{\partial x}x^3e^{xy} = 3x^2e^{xy} + x^3ye^{xy} = x^2e^{xy}(3+xy)$$
$$\frac{\partial}{\partial y}(2x+x^2y)e^{xy} = x^2e^{xy} + (2x+x^2y)xe^{xy} = x^2e^{xy}(3+xy)$$

These are equal, hence **F** is conservative and consequently all line integrals over closed paths C evaluate to zero.

What changes in three dimensions? Essentially nothing! A connected volume E is still simply-connected if every curve can be shrunk to a point without leaving E.



The interior of a solid torus E is non-simply-connected since a curve can be drawn inside it which cannot be shrunk to a point

The primary difference is that the corresponding Theorem is not quite as easy to use as the 2D version…

Theorem. *Suppose* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ *is a vector field on a simply-connected region* $E \subseteq \mathbb{R}^3$ *and suppose that* P, Q, R *have continuous partial derivatives. Then* **F** *is conservative iff*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \textit{and} \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \quad \textit{throughout } E.$$

The proof requires Stokes' Theorem and will be much easier to remember once we've introduced Curl. At the present it is not worth remembering.

Example Let $\mathbf{F}(x,y) = \begin{pmatrix} 2x+2y+6xyz+z \\ 2x+3x^2z \\ 3x^2y+x \end{pmatrix}$. Writing $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ we can laboriously check that $P_y = Q_x$, etc., to see that $\mathbf{F}$ is indeed conservative.
Writng $\mathbf{F} = \nabla f$ we can partially integrate:

$$\begin{aligned} f_x &= P = 2x + 2y + 6xyz + z \\ &\implies f(x,y,z) = x^2 + 2xy + 3x^2yz + xz + g(y,z) \\ f_y &= Q = 2x + 3x^2z \\ &\implies f(x,y,z) = 2xy + 3x^2yz + h(x,z) \\ f_z &= R = 3x^2y + x \\ &\implies f(x,y,z) = 3x^2yz + xz + j(x,y) \end{aligned}$$

for unknown functions g, h, j. Suitable choices yield

$$f(x,y,z) = x^2 + 2xy + 3x^2yz + xz$$

Does Simple-Connectedness Really Matter? Let us analyze the vector field $\mathbf{F} = \dfrac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$ and try to decide if it is conservative.

1. First compute the partial derivatives:

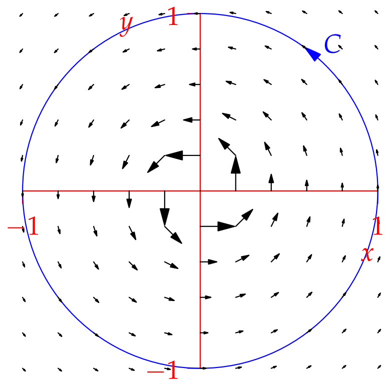
$$\begin{aligned} \frac{\partial Q}{\partial x} &= \frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}\frac{-y}{x^2+y^2} = \frac{-1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}$$

   Since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ we want to say that $\mathbf{F}$ is conservative.

2. Now calculate the line integral of $\mathbf{F}$ around the unit circle: $\mathbf{r}(t) = \left(\begin{smallmatrix} \cos t \\ \sin t \end{smallmatrix}\right)$, $0 \le t \le 2\pi$

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \mathrm{d}t = 2\pi$$

   Since $\int_C \mathbf{F} \cdot \mathrm{d}r \neq 0$ we conclude that $\mathbf{F}$ is non-conservative.

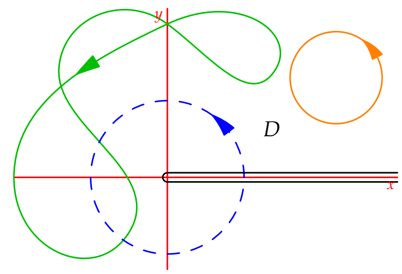


At least one of these arguments has to be false, but which? The answer, strangely, depends on your choice of *domain*.

- With no additional information, you should assume that the domain of the vector field is the largest set possible, in this case the punctured plane $D = \mathbb{R}^2 \setminus \{(0,0)\}$, the plane with the origin removed. This is non-simply-connected; indeed the unit circle cannot be shrunk to a point without some part of it passing through the origin! The easy $Q_x = P_y$ theorem does not apply and argument 1. is false. The vector field is conservative.

- Suppose that the domain was restricted so that it was simply connected. For instance, we could exclude the positive x-axis and choose the domain

$$D = \mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$$



Now D is simply-connected and $\mathbf{F}$ is conservative. Moreover $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve in D. Note that the original unit circle is no longer a curve in D, whence argument 2. is now false.

In the picture, the line integrals round the solid curves are zero, while around the blue curve (which is not in D) the line integral evaluates to 2π. You can easily check that a suitable potential function for $\mathbf{F}$ on D is $f = \theta$ in polar-coordinates; otherwise said,

$$f(x,y) = \begin{cases} \tan^{-1} \frac{y}{x} & y > 0 \\ \pi & y = 0 \\ \pi + \tan^{-1} \frac{y}{x} & y < 0 \end{cases}$$

The problem in extending this potential function to the entire punctured plane is that θ is not continuous everywhere; typically it is discontinuous on the positive x-axis.

Summary If $\mathbf{F}(x,y) = P\mathbf{i} + Q\mathbf{j}$ is defined on a connected region $D \subseteq \mathbb{R}^2$, where P, Q have continuous first derivatives then the following are equivalent:

1. $\mathbf{F}$ is conservative
2. $\mathbf{F}$ is a gradient field ($= \nabla f$ for some potential function f)
3. $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path for any curve C in D
4. $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every piecewise smooth closed curve C in D

In addition, if D is simply-connected we also have

5. $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ on a volume $E \subseteq \mathbb{R}^3$ then the fifth condition becomes

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$