16.4 Green's Theorem

Unless a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative, computing a line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y$$

can be difficult and time-consuming. For a given integral one must:

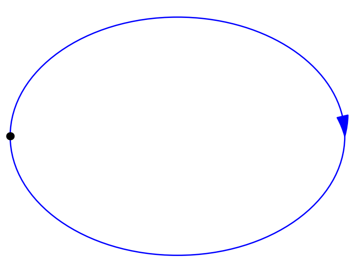
1. Split C into smooth sub-curves $C_1, C_2, \ldots$.
2. Parametrize each sub-curve by some vector-valued function $\mathbf{r}_i(t)$, $a_i \le t \le b_i$.
3. Evaluate each integral $\int_{C_i} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{a_i}^{b_i} \mathbf{F}\big(\mathbf{r}_i(t)\big) \cdot \mathbf{r}_i'(t)\,\mathrm{d}t$.

In this section we describe an important result for helping to evaluate and interpret line integrals along certain types of curve in the *plane*.[1]

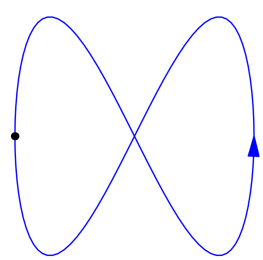
Definition. Recall that a curve C *closed* if it starts and finishes at the same point.

- A closed curve in the plane is *simple* if it has no self-intersections; it does not cross itself. This guarantees that the curve has a sensible *inside.*
- A simple closed curve in the plane is *positively-oriented* if the direction of travel keeps the inside of the curve on one's *left*. Alternatively: one traverses the curve *counter-clockwise.*

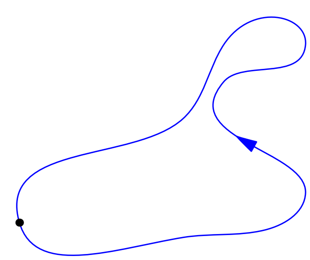
The notation $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ denotes a line integral around a *positively oriented, simple, closed curve* C.



A negatively-oriented simple closed curve



A non-simple closed curve



A positively-oriented simple closed curve

Positive-orientation fits with the right hand rule: curl the fingers of your right hand in the direction of the curve and your thumb will point out of the page.

Theorem (Green's Theorem)**.** *Let C be a positively-oriented, simple, closed, piecewise-smooth curve in the plane and denote its interior region by D. Suppose that P and Q have continuous partial derivatives on some open region containing D and its boundary C. Then*

$$\oint_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A = \iint_D Q_x - P_y\,\mathrm{d}A$$

If you're interested, a proof is at the end of this section.

[1]The corresponding result for curves in three dimensions, Stokes Theorem, will be addressed later.

Before seeing some examples, we tidy up our earlier discussion of simply-connected regions and conservative vector fields.

Corollary. *If* $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ *has continuous partial derivatives on a simply-connected region* D*, then* $\mathbf{F}$ *is conservative if and only if*

$$\mathbf{F} \text{ is conservative} \iff \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Proof. $(\Rightarrow)$ In the previous section we checked that this holds even when D is not simply-connected.

$(\Leftarrow)$ Let C be any simple closed curve in D. Since D is simply-connected, $\mathbf{F}$ satisfies the hypotheses of Green's Theorem on the interior $\tilde{D}$ of C. But then,

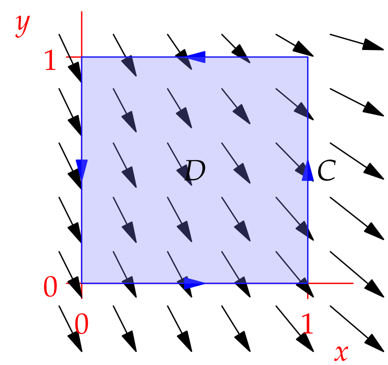
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_{\tilde{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = 0$$

We conclude that $\mathbf{F}$ is conservative.

■

Examples 1. Compute $\oint_C \mathbf{F}\cdot d\mathbf{r}$ around the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, where

$$\mathbf{F}(x,y) = (x^3+1)\mathbf{i} + (xy^2-2)\mathbf{j}$$



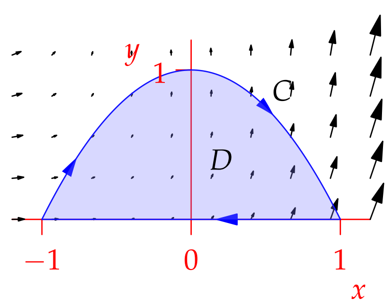
By Green's Theorem,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_R \frac{\partial}{\partial x}(xy^2-2) - \frac{\partial}{\partial y}(x^3+1)\,dA \\ &= \int_0^1\int_0^1 y^2\,dx\,dy = \frac{1}{3}\end{aligned}$$

We could check this by evaluating the line integral directly, though it would be tedious…

2. Calculate $\int_C (e^{x^2}+y)\,dx + (e^{2x}-y)\,dy$ where C is formed from the parabola $y = 1-x^2$ and the x-axis as drawn.

The orientation of C is negative, so Green's Theorem gets a minus sign:



$$\begin{aligned}\int_C \begin{pmatrix} e^{x^2}+y \\ e^{2x}-y\end{pmatrix}\cdot d\mathbf{r} &= -\iint_R \frac{\partial}{\partial x}(e^{2x}-y) - \frac{\partial}{\partial y}(e^{x^2}+y)\,dA \\ &= \int_{-1}^1\int_0^{1-x^2} 1-2e^{2x}\,dy\,dx \\ &= \int_{-1}^1 (1-x^2)(1-2e^{2x})\,dx \\ &= e^{2x}\left(x^2-x-\frac{1}{2}\right) + x - \frac{1}{3}x^3\Big|_{-1}^1 && \text{(integration by parts)} \\ &= \frac{4}{3} - \frac{1}{2}e^2 - \frac{3}{2}e^{-2}\end{aligned}$$

Calculating Areas

A powerful application of Green's Theorem is to find the area inside a curve:

Corollary. *If C is a positively oriented, simple, closed curve, then the area inside C is given by*

$$\oint_C x\,\mathrm{d}y = -\oint_C y\,\mathrm{d}x = \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x$$

Proof. Let D be the interior of C. By Green's Theorem,

$$\oint_C x\,\mathrm{d}y = \iint_D \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}0\,\mathrm{d}A = \iint_D \mathrm{d}A$$

The second integral is similar, while the third is the average of the first two. ∎

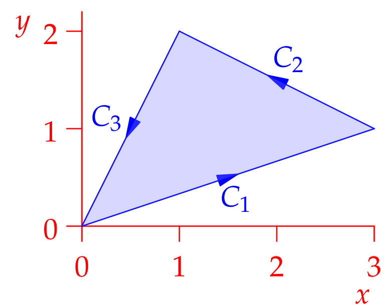
Examples 1. Find the area of the triangle with vertices $(0,0)$, $(3,1)$, $(1,2)$.

Parametrize the three sides, using $0 \le t \le 1$ each time:

$$\mathbf{r}_1(t) = t\begin{pmatrix}3\\1\end{pmatrix} \implies \mathrm{d}y = \mathrm{d}t$$

$$\mathbf{r}_2(t) = (1-t)\begin{pmatrix}3\\1\end{pmatrix} + t\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3-2t\\1+t\end{pmatrix} \implies \mathrm{d}y = \mathrm{d}t$$

$$\mathbf{r}_3(t) = (1-t)\begin{pmatrix}1\\2\end{pmatrix} \implies \mathrm{d}y = -2\,\mathrm{d}t$$



Now calculate:

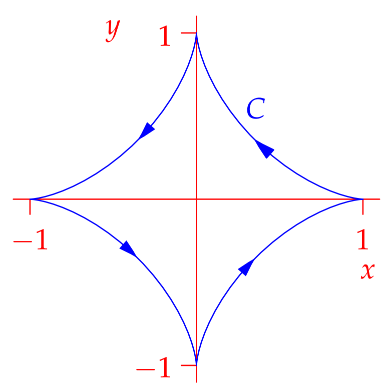
$$A = \oint_C x\mathrm{d}y = \int_0^1 3t\,\mathrm{d}t + (3-2t)\,\mathrm{d}t + (1-t)(-2\,\mathrm{d}t) = \int_0^1 3t + 1\,\mathrm{d}t = \frac{5}{2}$$

The same answer can also be found using basic geometry.

2. Find the area of the asteroid $x^{2/3} + y^{2/3} = 1$.

Parametrizing with $0 \le t \le 2\pi$, we have

$$\mathbf{r}(t) = \begin{pmatrix}\cos^3 t\\ \sin^3 t\end{pmatrix} \implies \begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix} = \begin{pmatrix}-3\sin t\cos^2 t\\ 3\cos t\sin^2 t\end{pmatrix}\mathrm{d}t$$



Therefore

$$\begin{aligned} A &= \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^{2\pi} \cos^3 t\cdot 3\cos t\sin^2 t - \sin^3 t(-3\sin t\cos^2 t)\,\mathrm{d}t \\ &= \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\,\mathrm{d}t = \frac{3}{8}\int_0^{2\pi} \sin^2 2t\,\mathrm{d}t = \frac{3}{16}\int_0^{2\pi} 1 - \cos 4t\,\mathrm{d}t = \frac{3}{8}\pi \end{aligned}$$

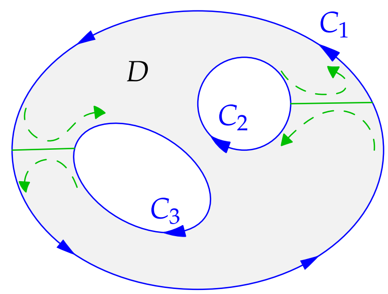
The $\frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x$ formula helped us take advantage of the identity $\cos^2 t + \sin^2 t = 1$.

Regions with holes

Green's Theorem is easily modified to apply to non-simply-connected regions.

In the picture, the boundary curve has three pieces $C = C_1 \cup C_2 \cup C_3$ oriented so that region D is always **on the left** of C.

If we join the curves by making two cuts (traversed in both directions), we recover Green's Theorem for a simply-connected region.

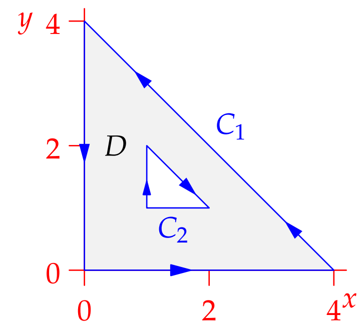


More generally, if $C = C_1 \cup C_2 \cup \cdots \cup C_n$ is the boundary of D, where all curves are oriented such that D is on the left, then

$$\int_C P\,dx + Q\,dy = \sum_{i=1}^{n} \int_{C_i} P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

Examples 1. Calculate the line integral $\int_C xy\,dx + dy$ where $C = C_1 \cup C_2$ is as pictured.

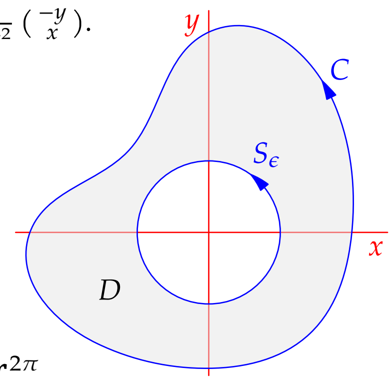
The pieces of C are oriented correctly, whence



$$\begin{aligned}\int_C xy\,dx + dy &= \iint_D -x\,dA \\ &= \int_0^4 \int_0^{4-x} -x\,dy\,dx - \int_1^2 \int_1^{3-x} -x\,dy\,dx \\ &= \int_0^4 x^2 - 4x\,dx + \int_1^2 2x - x^2\,dx = -10\end{aligned}$$

2. (*Winding Numbers*) Recall the rotational field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$.

Let C be *any* positively-oriented, simple, closed curve encircling the origin and draw a circle S_ε of radius ε *inside* C. By applying Green's Theorem to the region D between C and S_ε, we see that



$$\left(\oint_C - \oint_{S_\varepsilon}\right) \mathbf{F} \cdot d\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \iint_D 0\,dA = 0$$

Parametrizing S_ϵ via $\mathbf{r}(t) = \begin{pmatrix} \epsilon\cos t \\ \epsilon\sin t \end{pmatrix}$, we conclude that

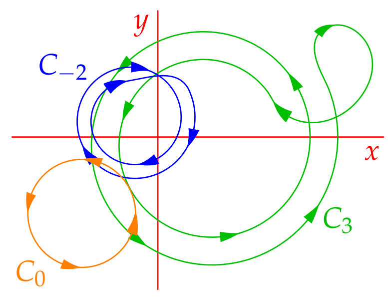
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{S_\varepsilon} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{1}{\epsilon^2} \begin{pmatrix} -\epsilon\sin t \\ \epsilon\cos t \end{pmatrix} \cdot \begin{pmatrix} \epsilon\cos t \\ \epsilon\sin t \end{pmatrix} dt = \int_0^{2\pi} dt = 2\pi$$

The value of the line integral round *any* such curve C always equals 2π!

More generally, if C is any closed curve avoiding the origin (it need not be simple!), then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \frac{x\,dy - y\,dx}{r^2} = 2\pi n$$

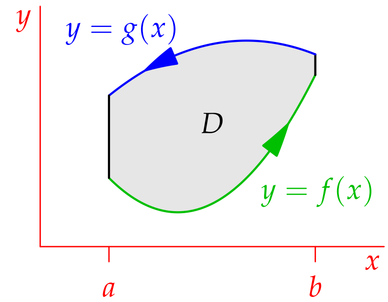
where n is the number of times C orbits the origin counter-clockwise. The integer n is called the *winding number* of C. The picture shows three curves: each C_n has winding number n.



A Sketch Proof of Green's Theorem

The proof has three stages. We prove half each of the theorem when D is a region of Type 1 or Type 2. The proof is completed by cutting up a general region into regions of both types.

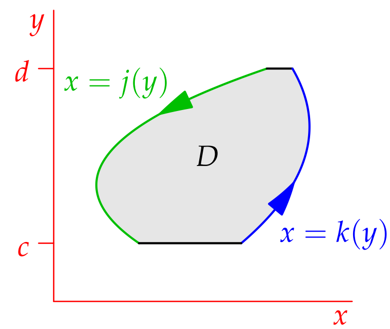
1. If D is a region of type 1, then its boundary C consists of 2–4 curves: $y = f(x)$ and $y = g(x)$ between $x = a$ and b and (possibly) two vertical edges at $x = a, b$.



$$\begin{aligned}\iint_D P_y \, dA &= \int_a^b \int_{f(x)}^{g(x)} P_y(x, y) \, dy \, dx \\ &= \int_a^b P(x, g(x)) - P(x, f(x)) \, dx \\ &= -\left(\int_a^b P(x, f(x)) \, dx + \int_b^a P(x, g(x)) \, dx \right) \\ &= -\oint_C P(x, y) \, dx\end{aligned}$$

where the final equality is because x is constant ($dx = 0$) on the vertical parts of C.

2. If D is a region of type 2, we proceed analogously.

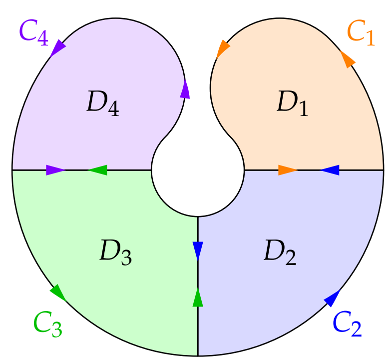


$$\begin{aligned}\iint_D Q_x \, dA &= \int_c^d \int_{j(y)}^{k(y)} Q_x(x, y) \, dx \, dy \\ &= \int_c^d Q(k(y), y) - Q(j(y), y) \, dy \\ &= \oint_C Q(x, y) \, dy\end{aligned}$$

This time $dy = 0$ on the horizontal parts of C.

3. If D is a region of *both* type 1 and 2, summing both parts gives the result:

$$\oint_C P \, dx + Q \, dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$



In general, D is cut into pieces of both types.[2] Line integrals along internal edges are taken in both directions and thus cancel. For example, the pictured $D = D_1 \cup D_2 \cup D_3 \cup D_4$ has been cut into four such regions. Note how the internal boundary between D_1 and D_2 is traversed in both directions. We conclude:

$$\begin{aligned}\oint_C P \, dx + Q \, dy &= \left[\oint_{C_1} + \cdots + \oint_{C_4} \right] P \, dx + Q \, dy \\ &= \left[\iint_{D_1} + \cdots + \iint_{D_4} \right] \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA\end{aligned}$$

[2] We will have to leave it as an article of faith that any suitable D may be so decomposed—study some topology!