16.4 Green's Theorem

Unless a vector field $\mathbf{F}$ is conservative, computing the line integral

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y$$

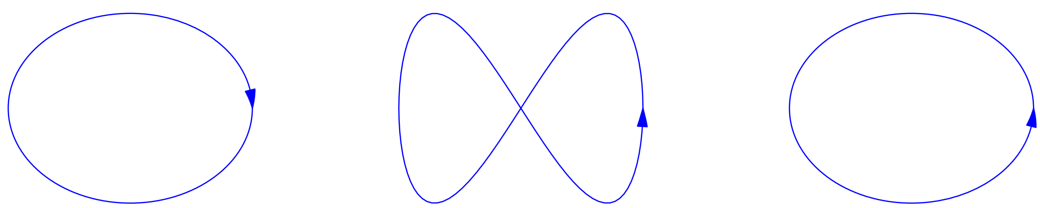
is often difficult and time-consuming. For a given integral one must:

1. Split C into separate smooth subcurves C_1, C_2, C_3.
2. Parameterize each curve C_i by a vector-valued function $\mathbf{r}_i(t),\ a_i \le t \le b_i$.
3. Evaluate each integral $\int_{C_i} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{a_i}^{b_i} \mathbf{F}(\mathbf{r}_i(t)) \cdot \mathbf{r}_i'(t)\,\mathrm{d}t$.

Thankfully there is a short-cut available for line integrals over particularly simple curves C.

Definition. *A curve C is the planes is:*

1. Closed *if it starts and finishes at the same point.*
2. Simple *if it has no self-intersections; it does not cross itself.*
3. Positively-oriented *if the direction of travel around C is such that the* inside *of C is on one's* left.

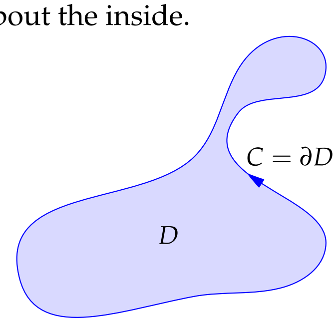


A negatively oriented simple closed curve | A non-simple closed curve | A positively oriented simple closed curve

Alternatively, a simple closed curve is positively oriented if one traverses it *counter-clockwise.* This categorization can easily lead you astray however, so it is better to think about the inside.

Definition. *The notation $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ denotes a line integral around a positively oriented, simple, closed curve C.*

If D is a region, then its boundary curve is denoted ∂D.



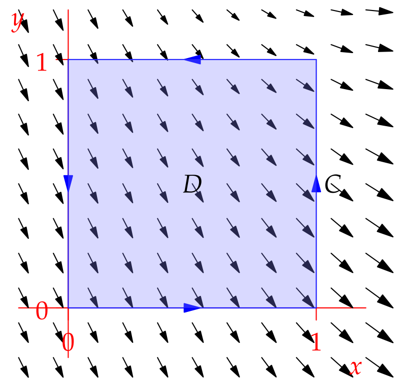
Observe that D is simply-connected iff its boundary ∂D is simple and closed.

Theorem (Green's Theorem)**.** *Let D be a simply-connected region of the plane with positively-oriented, simple, closed, piecewise-smooth boundary $C = \partial D$. Suppose that P, Q have continuous partial derivatives on some open region containing D and its boundary. Then*

$$\oint_C P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A = \iint_D Q_x - P_y\,\mathrm{d}A$$

Green's theorem is often useful in examples since double integrals are typically easier to evaluate than line integrals.

Example Find $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$, where C is the square with corners (0,0), (1,0), (1,1), (0,1), and



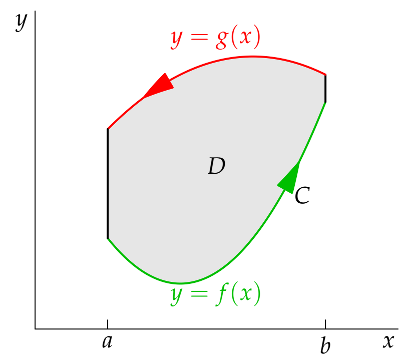
$$\mathbf{F}(x,y) = (x^3+1)\mathbf{i} + (xy^2-2)\mathbf{j}$$

By Green's Theorem,

$$\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \iint_R \frac{\partial}{\partial x}(xy^2-2) - \frac{\partial}{\partial y}(x^3+1)\,\mathrm{d}A$$
$$= \int_0^1\int_0^1 y^2\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{3}$$

We could check this by evaluating the line integral directly...

Proof of Green's Theorem. The proof has three stages. First prove half each of the theorem when the region D is either Type 1 or Type 2. Putting these together proves the theorem when D is both type 1 and 2. The proof is completed by cutting up a general region into regions of both types.



First suppose that R is a region of Type 1

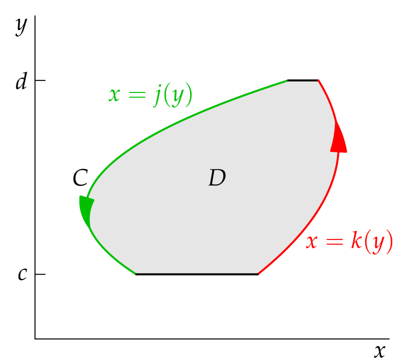
C consists of two, three, or four curves: $y = f(x)$ and $y = g(x)$ between $x = a$ and b and (possibly) two vertical edges at $x = a, b$. On the straight edges x is constant and so $\mathrm{d}x = 0$. Therefore

$$\oint_C P(x,y)\,\mathrm{d}x = \int_a^b P(x,f(x))\,\mathrm{d}x + \int_b^a P(x,g(x))\,\mathrm{d}x$$
$$= \int_a^b P(x,f(x)) - P(x,g(x))\,\mathrm{d}x$$

However

$$\iint_D P_y\,\mathrm{d}A = \int_a^b\int_{f(x)}^{g(x)} P_y(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_a^b P(x,g(x)) - P(x,f(x))\,\mathrm{d}x = -\oint_C P(x,y)\,\mathrm{d}x$$

Hence $\oint_C P(x,y)\,\mathrm{d}x = -\iint_D P_y\,\mathrm{d}A$ which is half of the theorem.



Now suppose that R is a region of Type 2

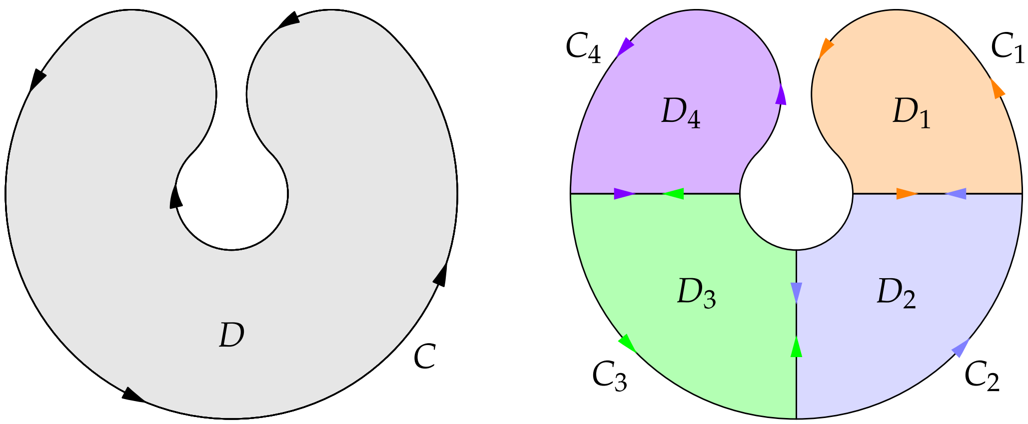
Analogously to before, we compute

$$\oint_C Q(x,y)\,\mathrm{d}y = \int_d^c Q(j(y),y)\,\mathrm{d}y + \int_c^d Q(k(y),y)\,\mathrm{d}y$$
$$= \int_c^d Q(k(y),y) - Q(j(y),y)\,\mathrm{d}y$$
$$= \int_c^d\int_{j(y)}^{k(y)} Q_x(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_R Q_x\,\mathrm{d}A$$

This is the other half of the theorem. It follows that if D is a region of *both* Types 1 & 2 we have the result:

$$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A$$

For general regions, cut D into pieces of both Types 1 & 2: line integrals along common edges are counted in both directions and thus cancel.[1]



For example, $D_1, \ldots, D_4$ are of Types 1 & 2 so Green's Theorem holds for each piece. Notice however that all boundaries are traversed twice in opposite directions. For example, the curves C_1 and C_2 run opposite to each other on the common boundary of D_1 and D_2. It follows that the line integrals along these pieces cancel each other out. Therefore

$$\begin{aligned}\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y &= \left[\oint_{C_1} + \cdots + \oint_{C_4}\right] P\,\mathrm{d}x + Q\,\mathrm{d}y \\ &= \left[\iint_{D_1} + \cdots + \iint_{D_4}\right] \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A\end{aligned}$$

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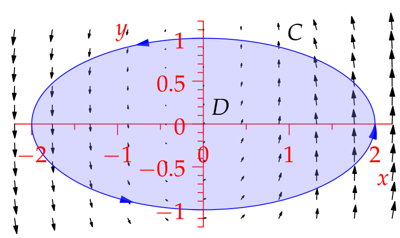
Examples

1. Let C be the perimeter of the ellipse



$$\frac{x^2}{4} + y^2 = 1$$

and

$$\mathbf{F}(x,y) = \begin{pmatrix} \sin(e^x) - y \\ 4x + \cos(y^2) \end{pmatrix}$$

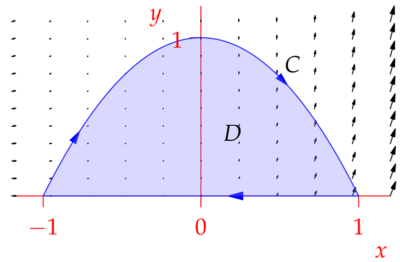
Then

$$\begin{aligned}\oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \iint_D \frac{\partial}{\partial x}(4x + \cos(y^2)) - \frac{\partial}{\partial y}(\sin(e^x) - y)\,\mathrm{d}A \\ &= \iint_D 4 + 1\,\mathrm{d}A = 5\iint_D \mathrm{d}A = 10\pi \qquad \text{(area of ellipse formula)}\end{aligned}$$

This is contrived, but it would have been much harder to evaluate directly as a line integral.

[1] We will have to leave it as an article of faith that any simply-connected region D may be so decomposed.

2. Calculate $\int_C (e^{x^2} + y)\,\mathrm{d}x + (e^{2x} - y)\,\mathrm{d}y$ where C is formed from the parabola $y = 1 - x^2$ and the x-axis as shown
The orientation of C is negative, so Green's Theorem gets a minus sign:



$$\begin{aligned}
\int_C \begin{pmatrix} e^{x^2}+y \\ e^{2x}-y \end{pmatrix} \cdot \mathrm{d}\mathbf{r} &= -\iint_R \frac{\partial}{\partial x}(e^{2x} - y) - \frac{\partial}{\partial y}(e^{x^2} + y)\,\mathrm{d}A \\
&= \int_{-1}^{1} \int_0^{1-x^2} 1 - 2e^{2x}\,\mathrm{d}y\,\mathrm{d}x = \int_{-1}^{1} (1 - x^2)(1 - 2e^{2x})\,\mathrm{d}x \\
&= e^{2x}\left(x^2 - x - \frac{1}{2}\right) + x - \frac{1}{3}x^3 \Big|_{-1}^{1} \qquad \text{(integration by parts)} \\
&= \frac{4}{3} - \frac{1}{2}e^2 - \frac{3}{2}e^{-2}
\end{aligned}$$

Simple-connectedness revisited We are now in a position to prove our simple formula: if $\mathbf{F}$ has continuous partial derivatives on a simply-connected region D, then

$$\mathbf{F} = P\mathbf{i} + Q\mathbf{j} \text{ conservative } \iff \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Proof. Recall that $\mathbf{F}$ is conservative on D if and only if $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for all C in D.
D simply-connected $\implies$ interior $\widetilde{D}$ of every simple-closed curve C in D is also simply-connected.
By Green's Theorem,

$$\mathbf{F} \text{ conservative } \iff 0 = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_{\widetilde{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A$$

for all such curves C. This says that $\iint_{\widetilde{D}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A = 0$ *independent of the domain* $\widetilde{D}$. This is only possible if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ everywhere. ∎

Calculating Areas A powerful application of Green's Theorem is to find the area inside a curve:

Theorem. *If C is a positively oriented, simple, closed curve, then the area inside C is given by*

$$\oint_C x\,\mathrm{d}y = -\oint_C y\,\mathrm{d}x = \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x$$

Proof. If D the interior of C then, by Green's Theorem,

$$\begin{aligned}
&\oint_C x\,\mathrm{d}y = \iint_D \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}0\,\mathrm{d}A = \iint_D \mathrm{d}A, \quad \text{and,} \\
&-\oint_C y\,\mathrm{d}x = -\iint_D \frac{\partial}{\partial x}0 - \frac{\partial}{\partial y}y\,\mathrm{d}A = \iint_D \mathrm{d}A
\end{aligned}$$

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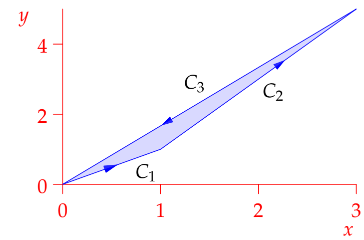
Examples

1. Find the area of the triangle with vertices (0,0), (1,1), (3,5)



   Parameterizing with $0 \le t \le 1$ each time, the triangle has three parts

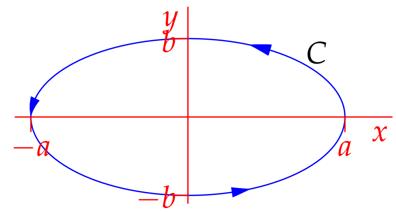
$$C_1: \ \mathbf{r}_1(t) = t\begin{pmatrix}1\\1\end{pmatrix} \implies \mathrm{d}y = \mathrm{d}t$$
$$C_2: \ \mathbf{r}_2(t) = t\begin{pmatrix}3\\5\end{pmatrix} + (1-t)\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1+2t\\1+4t\end{pmatrix} \implies \mathrm{d}y = 4\,\mathrm{d}t$$
$$C_3: \ \mathbf{r}_3(t) = (1-t)\begin{pmatrix}3\\5\end{pmatrix} \implies \mathrm{d}y = -5\,\mathrm{d}t$$

   We thus calculate:

$$A = \oint_C x\,\mathrm{d}y = \int_0^1 t + 4(3t+1-t) - 5(3(1-t))\,\mathrm{d}t = \int_0^1 24t - 11\,\mathrm{d}t = 1$$

   The same answer can be found more easily using basic geometry.

2. Use Green's Theorem to find the area of the ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

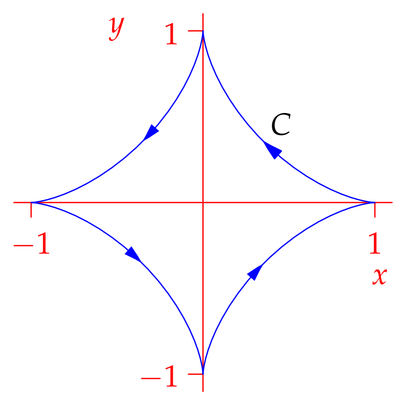
   Parameterizing with $0 \le t \le 2\pi$, we have

$$C: \ \mathbf{r}(t) = \begin{pmatrix}a\cos t\\ b\sin t\end{pmatrix} \implies \begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix} = \begin{pmatrix}-a\sin t\\ b\cos t\end{pmatrix}\mathrm{d}t$$

   Therefore

$$A = \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x = \frac{1}{2}\int_0^{2\pi} a\cos t \cdot b\cos t - b\sin t(-a\sin t)\,\mathrm{d}t$$
$$= \frac{1}{2}\int_0^{2\pi} ab\,\mathrm{d}t = \pi ab$$

3. Find the area of the asteroid



$$x^{2/3} + y^{2/3} = 1$$

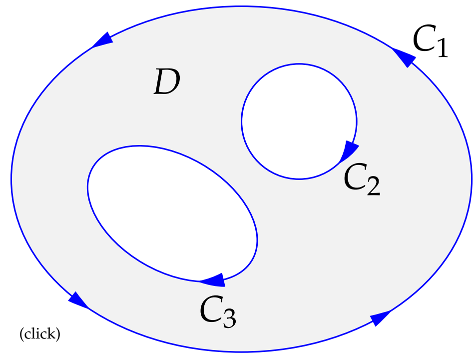
   Parameterizing with $0 \le t \le 2\pi$, we have

$$C: \ \mathbf{r}(t) = \begin{pmatrix}\cos^3 t\\ \sin^3 t\end{pmatrix} \implies \begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\end{pmatrix} = \begin{pmatrix}-3\sin t\cos^2 t\\ 3\cos t\sin^2 t\end{pmatrix}\mathrm{d}t$$

   Therefore

$$A = \frac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x = \frac{1}{2}\int_0^{2\pi} \cos^3 t \cdot 3\cos t\sin^2 t - \sin^3 t(-3\sin t\cos^2 t)\,\mathrm{d}t$$
$$= \frac{3}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\,\mathrm{d}t = \frac{3}{8}\int_0^{2\pi} \sin^2 2t\,\mathrm{d}t$$
$$= \frac{3}{16}\int_0^{2\pi} 1 - \cos 4t\,\mathrm{d}t = \frac{3}{8}\pi$$

Regions with holes Green's Theorem can be modified to apply to non-simply-connected regions.

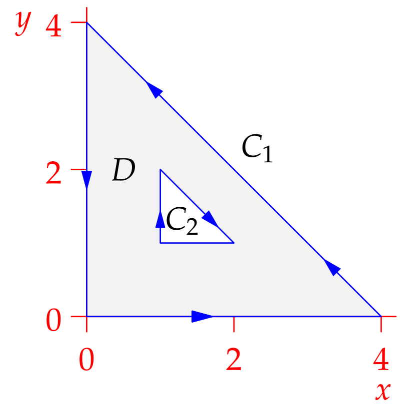


In the picture, the boundary curve has three pieces $C = C_1 \cup C_2 \cup C_3$ oriented so that region D is always on the left of the boundary.
Join the curves together with cuts (traversed both directions) to recover Green's Theorem for a simply-connected region

If the boundary of D is made up of n curves $C = C_1 \cup C_2 \cup \cdots \cup C_n$ all oriented so that D is on the left, then

$$\int_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \sum_{i=1}^{n} \int_{C_i} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A$$

Example Calculate the line integral $\int_C xy\,\mathrm{d}x + \mathrm{d}y$ where $C = C_1 \cup C_2$ is the curve shown.

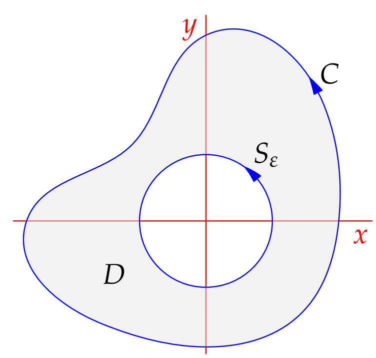


The pieces of C are oriented correctly for Green's Theorem:

$$\begin{aligned}\int_C xy\,\mathrm{d}x + \mathrm{d}y &= \iint_R -x\,\mathrm{d}A \\ &= \int_0^4 \int_0^{4-x} -x\,\mathrm{d}y\,\mathrm{d}x - \int_1^2 \int_1^{3-x} -x\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^4 x^2 - 4x\,\mathrm{d}x + \int_1^2 2x - x^2\,\mathrm{d}x = -10\end{aligned}$$

Winding Numbers Finally we return to our rotational vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$ from the previous section. We can use Green's Theorem to compute the line integral of $\mathbf{F}$ around *any* positively oriented, simple, closed curve surrounding the origin.

Let C be any such curve. Draw a circle S_ε of small radius ε inside C. Apply Green's Theorem to the region D between C and S_ε, noting that S_ε is oriented incorrectly for the theorem:



$$\left(\oint_C - \oint_{S_\varepsilon}\right) \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}A = 0$$

since $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ everywhere. It follows that

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_{S_\varepsilon} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

for all curves C.

Parameterizing S_ε by $\mathbf{r}(t) = \varepsilon \left(\begin{smallmatrix} \cos t \\ \sin t \end{smallmatrix} \right)$, $0 \le t \le 2\pi$ gives

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \oint_{S_\varepsilon} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \frac{1}{\varepsilon^2} \begin{pmatrix} -\varepsilon \sin t \\ \varepsilon \cos t \end{pmatrix} \cdot \begin{pmatrix} -\varepsilon \sin t \\ \varepsilon \cos t \end{pmatrix} \mathrm{d}t = \int_0^{2\pi} \mathrm{d}t = 2\pi$$

The line integral around all such curves C is therefore 2π.

We already know that if C is closed and does not orbit the origin O, then $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$. It follows that if C is any simple closed curve in the plane which avoids the O, then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \frac{1}{x^2 + y^2} (x\,\mathrm{d}y - y\,\mathrm{d}x) = 2\pi n$$

where n is the number of times C orbits O counter-clockwise. The integer $n = \frac{1}{2\pi} \int_C \frac{1}{r^2} (x\,\mathrm{d}y - y\,\mathrm{d}x)$ is called the *winding number* of C, and is an important concept in topology.
For example the orange curve has winding number 3, while the blue has winding number -2.