16.5 Curl and Divergence

Curl and *divergence* are two ways to differentiate a vector field that have useful physical Their behavior and notation is similar to that of cross and dot products. To see this, it is necessary to recall the differential operator *nabla* (∇) and the gradient of a scalar function f:

$$\operatorname{grad} f = \nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Definition. The *curl* of a differentiable vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the *vector field*

$$\begin{aligned}\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\ &= \begin{vmatrix}\partial_y & \partial_z \\ Q & R\end{vmatrix}\mathbf{i} - \begin{vmatrix}\partial_x & \partial_z \\ P & R\end{vmatrix}\mathbf{j} + \begin{vmatrix}\partial_x & \partial_y \\ P & Q\end{vmatrix}\mathbf{k}\end{aligned}$$

A vector field whose curl is everywhere zero is termed *irrotational.*

If desired, a two-dimensional vector field can be treated by taking $R = 0$: its curl is $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$.

Either notation ($\operatorname{curl} \mathbf{F}$ or $\nabla \times \mathbf{F}$) can be used. The cross-product construction is typically easier to remember than the formula.

Examples 1. $\operatorname{curl}\begin{pmatrix} x \\ y \end{pmatrix} = \nabla \times (x\mathbf{i} + y\mathbf{j}) = \begin{pmatrix}\partial_x \\ \partial_y \\ \partial_z\end{pmatrix} \times \begin{pmatrix} x \\ y \\ 0\end{pmatrix} = \mathbf{0}.$

2. $\operatorname{curl}\begin{pmatrix} -y \\ x \end{pmatrix} = \nabla \times (-y\mathbf{i} + x\mathbf{j}) = \begin{pmatrix}\partial_x \\ \partial_y \\ \partial_z\end{pmatrix} \times \begin{pmatrix} -y \\ x \\ 0\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1+1\end{pmatrix} = 2\mathbf{k}.$

3. If $\mathbf{F} = (x^2 z - 3y)\,\mathbf{i} + e^x\,\mathbf{j} + \sin(x + yz)\,\mathbf{k}$, then

$$\nabla \times \mathbf{F} = \begin{pmatrix}\partial_x \\ \partial_y \\ \partial_z\end{pmatrix} \times \begin{pmatrix} x^2 z - 3y \\ e^x \\ \sin(x+yz)\end{pmatrix} = \begin{pmatrix} z\cos(x+yz) - 0 \\ -\cos(x+yz) + x^2 \\ e^x + 3\end{pmatrix} = \begin{pmatrix} z\cos(x+yz) \\ x^2 - \cos(x+yz) \\ e^x + 3\end{pmatrix}$$

Curl and Conservatism The three components of curl should seem familiar. Revisiting section 16.3, observe that a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ with continuous partial derivatives on a simply-connected region is conservative if and only

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$

This says precisely that all parts of the curl vanish.

Theorem. *Suppose* $\mathbf{F}$ *has continuous first-derivatives. Then*:

1. *If* $\mathbf{F}$ *is conservative, then* $\operatorname{curl} \mathbf{F} = \mathbf{0}$ (*alternatively* $\nabla \times \nabla f = \mathbf{0}$—*conservative fields are irrotational*).

2. *If the domain of* $\mathbf{F}$ *is simply-connected and* $\operatorname{curl} \mathbf{F} = \mathbf{0}$, *then* $\mathbf{F}$ *is conservative.*

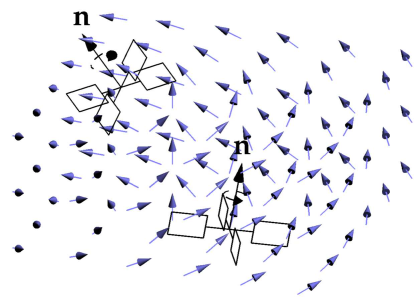
The Meaning of Curl: Local Rotation

As the term *irrotational* ($\operatorname{curl} \mathbf{F} = \mathbf{0}$) suggests, curl measures the tendency of objects to *rotate*.

Theorem. *Place an infinitesimal paddle with unit vector axis* $\mathbf{n}$ *in a fluid with velocity field* $\mathbf{F}$. *Assuming no friction, the paddle will rotate with angular speed*

$$\omega = \frac{1}{2}\mathbf{n} \cdot \operatorname{curl} \mathbf{F} \quad \textit{rad/s}$$

Otherwise said, the angular velocity of the paddle is half the projection of the curl onto the paddle's axis: $\boldsymbol{\omega} = \frac{1}{2}(\mathbf{n} \cdot \operatorname{curl} \mathbf{F})\mathbf{n} = \frac{1}{2} \operatorname{proj}_{\mathbf{n}}(\operatorname{curl} \mathbf{F})$.

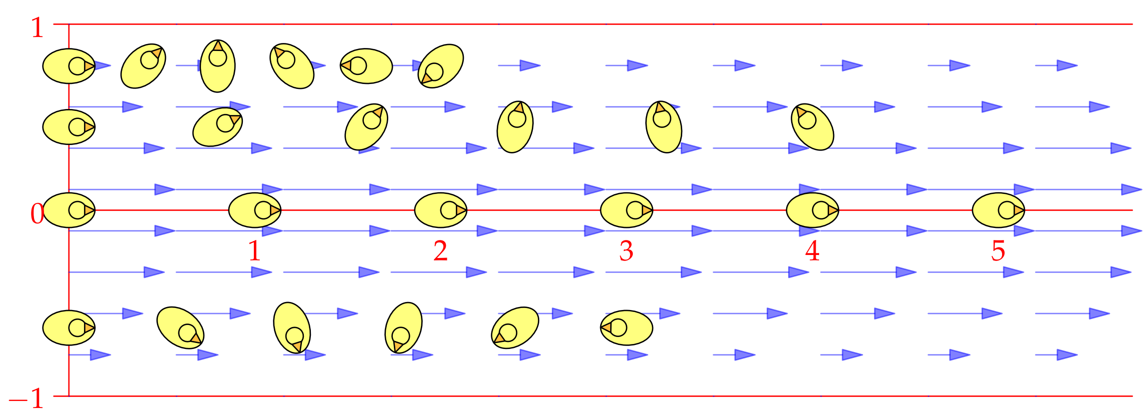


The proof is a little tricky and (at least in three dimensions) requires Stokes Theorem.

Duck Races! To help understand local rotation, it is helpful to imagine ducks racing down a stream. Suppose that the water flowing in a stream of width 2 units has velocity field $\mathbf{v} = (1 - y^2)\mathbf{i}$ (units/s). The curl of this flow is

$$\nabla \times \mathbf{v} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} 1 - y^2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2y \end{pmatrix} = 2y\,\mathbf{k}$$

Suppose we release several ducks[1] into the stream and that they move with the water. In the picture, four ducks are released from the starting line and their locations and orientations plotted every second.



There are two aspects to the motion:

Linear The ducks follow the direction of flow to the right *in straight lines*. The midstream duck is fastest with linear speed $v = 1$.

Rotational The midstream side of a duck is pushed rightwards more than the side closer to the stream's edge; this causes most ducks to rotate. According to the Theorem, a duck at $y = \frac{1}{2}$ rotates with angular velocity $\frac{1}{2}\nabla \times \mathbf{v} = \frac{1}{2}\mathbf{k}$ (counter-clockwise at $\frac{1}{2}$ rad/28.6° per second).

If you've ever been white-water rafting this explains why you should stay in the middle of the river: you travel fastest and you don't spin out of control!

[1] Mathematically, the ducks should be infinitesimal, but that would be no fun. An animated version is linked online.

Global Rotation $\neq$ Local Rotation It is important to distinguish between *local rotation* (changing the direction an object faces) and *global rotation* (vector fields making objects travel in loops). In the following three examples, the ducks follow circular trajectories counter-clockwise around the origin, but their local rotations are different.

All are specific cases of a general family of vector fields: $\mathbf{F}_n(x, y) = \frac{1}{r^n}\binom{-y}{x}$

Streamlines The trajectories are circles centered at the origin.[2]

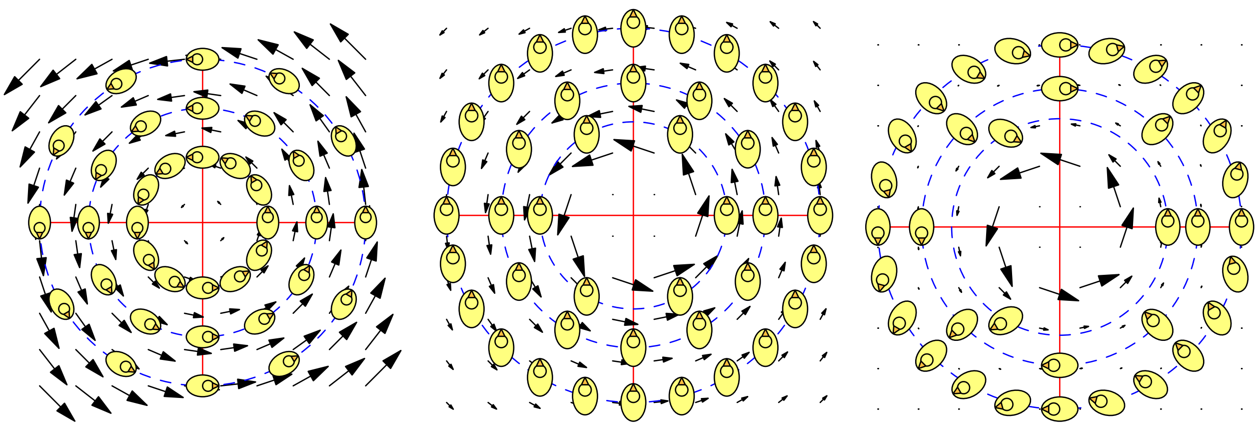
Global rotation The linear speed is $|\mathbf{F}_n| = r^{1-n}$. The global angular velocity around the circular trajectories is $\omega_g = \frac{r^{1-n}}{r} = r^{-n}$ rad/s.

Local Rotation $\operatorname{curl}\mathbf{F}_n = (2-n)r^{-n}\mathbf{k}$. The local rotational angular velocity is $\omega_l = \frac{2-n}{2}r^{-n}$ rad/s.

1. $\mathbf{F}_0 = \binom{-y}{x}$: The global and local angular rotations are equal $\omega_g = \omega_l = 1$ rad/s, with all ducks taking 2π seconds to complete an orbit. The vector field pushes more on the outside of each duck, rotating the direction it faces at the same rate as it rotates around the origin.

2. $\mathbf{F}_2 = \frac{1}{r^2}\binom{-y}{x}$: The global rotation is faster nearer the origin $\omega_g = r^{-2}$ rad/s. The local angular rotation is $\omega_l = 0$. The ducks always face their starting direction. The infinitesimal part is a little harder to visualize: the arrows on the outside of each duck are shorter, but slightly more of each duck counts as being "on the outside," so the local rotational effect balances out.

3. $\mathbf{F}_4 = \frac{1}{r^4}\binom{-y}{x}$: The global rotation $\omega_g = r^{-4}$ rad/s is much faster nearer the origin. The local rotation $\omega_l = -r^{-4}$ rad/s is *negative* the global. The ducks' local rotation is *clockwise*, since the arrows on the inside of each duck are so much stronger than those on the outside.



$\mathbf{F}_0 = \binom{-y}{x}$, $\nabla \times \mathbf{F}_0 = 2\mathbf{k}$; $\omega_l = 1$ counter-clockwise

$\mathbf{F}_2 = \frac{1}{r^2}\binom{-y}{x}$, $\nabla \times \mathbf{F}_2 = \mathbf{0}$; $\omega_l = 0$ rad/s

$\mathbf{F}_4 = \frac{1}{r^4}\binom{-y}{x}$, $\nabla \times \mathbf{F}_4 = -\frac{2}{r^4}\mathbf{k}$; $\omega_l = -\frac{1}{r^4}$ rad/s *clockwise*

The outer circular trajectories have radius 1, and the ducks on each trajectory are plotted at equally separated times ($\Delta t = \frac{\pi}{6}$ seconds in the first pictures and $\Delta t = \frac{\pi}{12}$ in the others). Animated versions of all the duck pictures can be found online.

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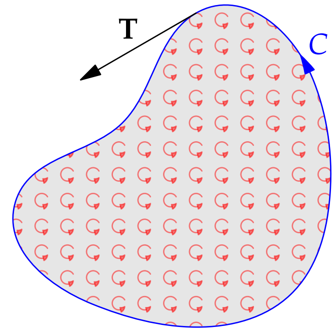
[2]If you've studied differential equations, these come from solving the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{r^n}\begin{pmatrix} -y \\ x \end{pmatrix} \implies \int y\,\mathrm{d}y = -\int x\,\mathrm{d}x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} \implies x^2 + y^2 = C$$

Circulation: Green's Theorem Revisited



If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then the expression $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ forms part of Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

Recall that $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds$ where $\mathbf{T}$ is the unit tangent vector of C. This line integral is called the *circulation* of the vector field $\mathbf{F}$ around C: it measures how vigorously $\mathbf{F}$ pushes a particle around the curve.
Interpreting curl as local rotation, we can reinterpret Green's Theorem: summing the infinitesimal rotations *inside* C equals the total circulation *around* C.

Divergence

Whereas curl describes the local rotation of a vector field, our second important derivative describes the tendency of a vector field to cause local expansion or contraction.

Definition. The *divergence* of a differentiable vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is the *scalar* function

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \\ R \end{pmatrix} \qquad (R = 0 \text{ in two dimensions})$$

A vector field whose divergence is everywhere zero is termed *incompressible*.
The *Laplacian* of a scalar function $f(x, y, z)$ is the divergence of its gradient:

$$\nabla^2 f := \nabla \cdot \nabla f = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial z \end{pmatrix} \cdot \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = f_{xx} + f_{yy} + f_{zz} \qquad (f_{xx} + f_{yy} \text{ in two dimensions})$$

Examples 1. $\operatorname{div}\big(x\mathbf{i} + (2x + y)\mathbf{j} + (3xyz^2)\mathbf{k}\big) = 1 + 1 + 6xyz$

2. $\nabla^2(x^2 - y^2) = \nabla \cdot \left(\begin{smallmatrix} 2x \\ -2y \end{smallmatrix}\right) = 2 - 2 = 0$. Otherwise said, $f(x, y) = x^2 - y^2$ is a solution to *Laplace's equation* $\nabla^2 f = 0$.

Laplacians have many applications, particularly in Physics. All such applications are matters for other courses. For instance, it appears in two famous partial differential equations:

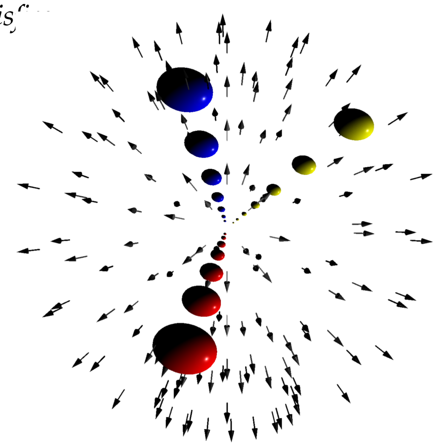
The Heat Equation $\frac{\partial T}{\partial t} = k\nabla^2 T = k(T_{xx} + T_{yy} + T_{zz})$ ($k > 0$ constant) models the temperature of a medium at position (x, y, z) and time t.

The Wave Equation $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$ models the amplitude with speed c.

The Meaning of Divergence: Local Expansion

As with curl and local rotation, the main result here is a little technical.

Theorem. *Suppose an infinitesimal balloon drifts along a path according to a velocity field* **F**. *If the surface of the balloon also moves with the vector field, then the balloon's volume satisfies*

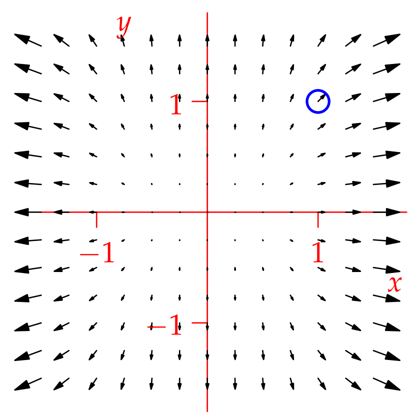


$$\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t} = \operatorname{div}\mathbf{F}$$

The larger the divergence, the greater the rate of volume increase. For a two-dimensional vector field, replace the balloon with a tiny oil drop: divergence is the logarithmic derivative of area. A proof will have to wait until we have the divergence theorem.

Example 1 The vector field $\mathbf{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{r}$ has constant divergence 3. An infinitesimal balloon released into the field will see its volume increase since $V'(t) = 3V(t) > 0$.

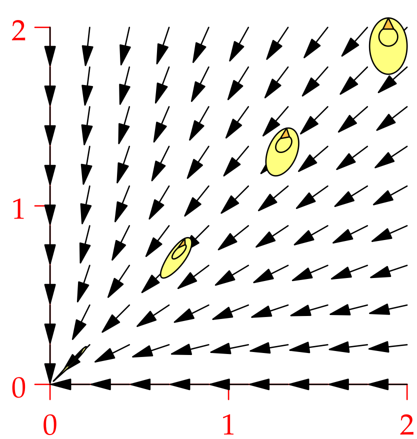
Example 2 The vector field $\mathbf{F} = \begin{pmatrix} x^3 \\ y \end{pmatrix}$ has divergence



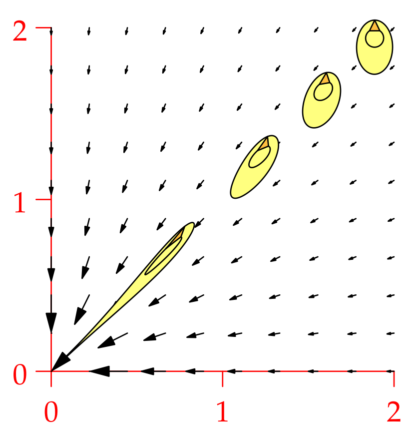
$$\nabla \cdot \begin{pmatrix} x^3 \\ y \end{pmatrix} = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \cdot \begin{pmatrix} x^3 \\ y \end{pmatrix} = \frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial y}y$$
$$= 3x^2 + 1$$

The fact that the divergence is everywhere *positive* indicates that the vector field tends to cause *areas to increase*. Try to visualize what happens to the small circle in the picture if its edges are allowed to drift with the field. Since the arrows on its upper-right side are the longest, its shape would elongate in that direction…

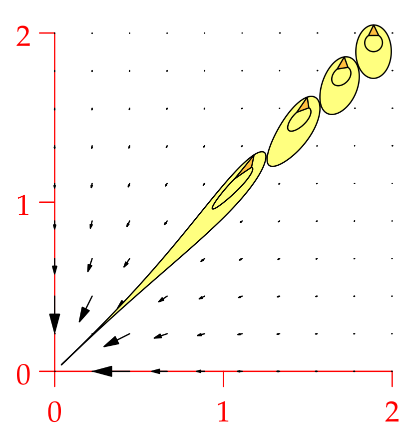
Global Expansion ≠ Local Expansion In the following pictures/animations, pretend that each duck is an oil-drop drifting with the field. While each duck drifts towards the origin (global contraction), the sign of the divergence affects the area of the duck.



$\mathbf{F}_1 = -\frac{1}{r}\begin{pmatrix} x \\ y \end{pmatrix}$, $\nabla \cdot \mathbf{F}_1 = -\frac{1}{r} < 0$
area decreases



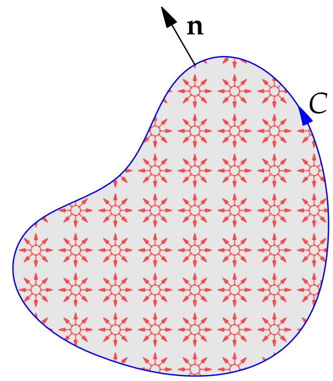
$\mathbf{F}_2 = -\frac{1}{r^2}\begin{pmatrix} x \\ y \end{pmatrix}$, $\nabla \cdot \mathbf{F}_2 = 0$
area constant



$\mathbf{F}_3 = -\frac{1}{r^3}\begin{pmatrix} x \\ y \end{pmatrix}$, $\nabla \cdot \mathbf{F}_3 = \frac{1}{r^3} > 0$
area increases

Green's Theorem, version III Suppose that $\mathbf{r}(t) = \binom{x(t)}{y(t)}$, $a \le t \le b$ parametrizes a positively-oriented, simple, closed curve. Its *outward pointing normal vector* $\mathbf{n}$ is obtained by rotating the unit tangent vector $\mathbf{T} = \frac{1}{|\mathbf{r}'|}\mathbf{r}'$ by 90° clockwise, that is,

$$\mathbf{n}(t) = \frac{1}{|\mathbf{r}'(t)|}\begin{pmatrix} y'(t) \\ -x'(t) \end{pmatrix}$$



We compute a new line integral, and convert to a double integral using Green's Theorem.

$$\begin{aligned} \oint_C \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s &= \int_a^b \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \frac{1}{|\mathbf{r}'|}\begin{pmatrix} y' \\ -x' \end{pmatrix} |\mathbf{r}'|\,\mathrm{d}t \\ &= \oint_C -Q\,\mathrm{d}x + P\,\mathrm{d}y \\ &= \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\,\mathrm{d}A = \iint_D \operatorname{div}\mathbf{F}\,\mathrm{d}A \end{aligned}$$

Since $\mathbf{F}\cdot\mathbf{n}$ measures the component of $\mathbf{F}$ pointing across the boundary C, the line integral $\oint_C \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s$ measures the total flow rate of $\mathbf{F}$ out of the region D: this is called the *flux* of the vector field $\mathbf{F}$ across C.

In this modified context, Green's Theorem has a new interpretation: the sum of the infinitesimal expansions inside C equals the flux across C. This is precisely the Divergence Theorem in 2-dimensions.

Summary

Nabla $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ is a differential operator.

Gradient $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$ is a *vector field.*

Measures the direction and magnitude of the greatest increase in f.

Curl $\nabla\times\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$ is a *vector field.*

Measures local rotation of $\mathbf{F}$.

$\frac{1}{2}\operatorname{curl}\mathbf{F}\cdot\mathbf{n}$ is the angular speed of induced rotation with axis $\mathbf{n}$.

Divergence $\nabla\cdot\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ is a *scalar field.*

Measures the local (logarithmic) rate of change of volume.

$\operatorname{div}\mathbf{F} > 0 \implies$ local expansion, $\quad \operatorname{div}\mathbf{F} < 0 \implies$ local contraction.