16.5   Curl and Divergence

Many ways to differentiate a vector field $\mathbf{F}(x, y, z) = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix}$. The *nine* scalar 1st-derivatives form the *total derivative* or *Jacobian matrix*:

$$D\mathbf{F} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial z} \\ \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \end{pmatrix}$$

Entries of $D\mathbf{F}$ can combined in various ways: *curl* and *divergence* are two such combinations that have useful physical interpretations.

Curl

Definition. *If* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ *is a three-dimensional vector field then the* curl *of* $\mathbf{F}$ *is the* vector field

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

defined wherever all partial derivatives exist.[1]

Differential Operators and Notation   'Nabla' or 'Del' is the differential operator

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

The gradient of f is the action of the operator ∇ on f:

$$\operatorname{grad} f = \nabla f = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

The cross product notation for curl now makes sense:

$$\nabla \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \partial_y & \partial_z \\ Q & R \end{vmatrix} \mathbf{i} - \begin{vmatrix} \partial_x & \partial_z \\ P & R \end{vmatrix} \mathbf{j} + \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} \mathbf{k} = \cdots$$

This construction is easier to remember than the formula in the definition and is most simple in column-vector notation.

Example   If $\mathbf{F} = (x^2 - 3y)\mathbf{i} + xz\mathbf{j} + (x + yz)\mathbf{k}$ then

$$\nabla \times \mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} x^2 - 3y \\ xz \\ x + yz \end{pmatrix} = \begin{pmatrix} z - x \\ -1 \\ z + 3 \end{pmatrix}$$

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[1]Some authors will treat two-dimensional vector fields and state that if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is such then its curl is the scalar quantity $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, familiar from Green's Theorem.

Curl and Conservatism Recall Section 16.3 where a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ on a simply-connected region is conservative if and only if

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$$
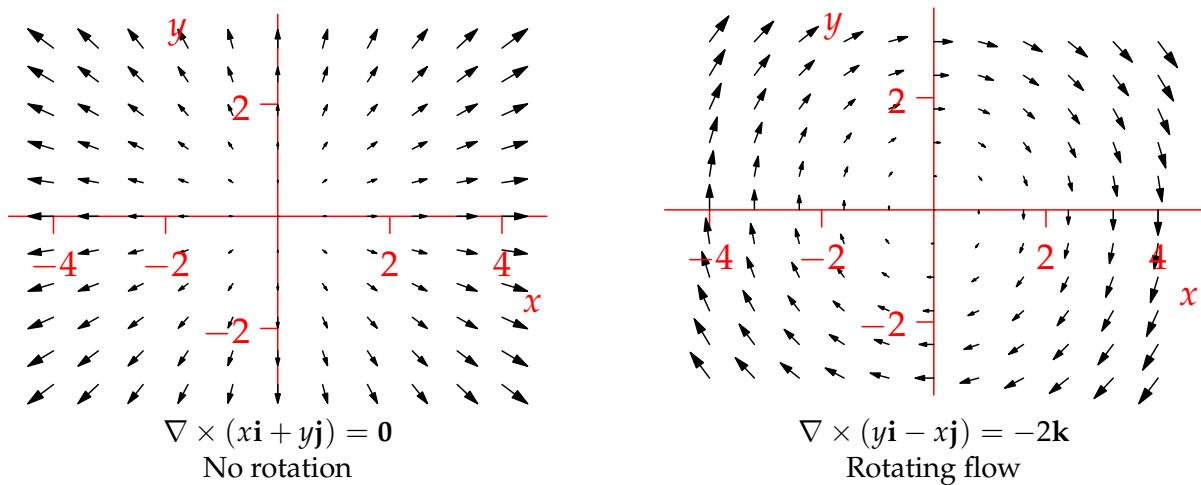
This says precisely that all parts of the curl vanish.

Theorem. *If $\mathbf{F}$ has continuous first-derivatives on a simply-connected region of $\mathbb{R}^3$, then* $\operatorname{curl}\mathbf{F} = \mathbf{0} \iff \mathbf{F}$ *is conservative*

One direction of this theorem can be written as follows:

Corollary. *If f has continuous second-derivatives, then* $\operatorname{curl}(\nabla f) = \mathbf{0}$ *   (often written $\nabla \times \nabla f = \mathbf{0}$).*

Rotating fields: Interpretation of Curl Curl measures the tendency of objects to *rotate*.

$$\nabla \times (x\mathbf{i} + y\mathbf{j}) = \mathbf{0}$$
No rotation

$$\nabla \times (y\mathbf{i} - x\mathbf{j}) = -2\mathbf{k}$$
Rotating flow

Definition. *A vector field $\mathbf{F}$ is said to be* irrotational *if $\operatorname{curl}\mathbf{F} = \mathbf{0}$ everywhere.*
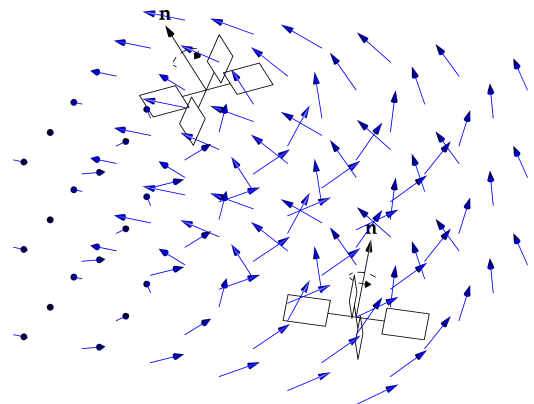
Local rotation The picture is too simplistic: curl measures *local* rotation. The fact that an object in the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ will travel in circles round the origin is irrelevant to curl. It is the fact that the object will also rotate so that it *changes the direction it is facing.*

Theorem. *Suppose that a small paddle with vertical axis is positioned at the point (a, b) in a fluid with velocity field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Assuming no friction and that the paddle rotates freely, the paddle will rotate with angular velocity*

$$\boldsymbol{\omega} = \frac{1}{2}\left(\frac{\partial Q}{\partial x}(a, b) - \frac{\partial P}{\partial y}(a, b)\right)\mathbf{k}$$

More generally, place a paddle at position $\mathbf{r}$ and with axis $\mathbf{n}$ in a fluid flow $\mathbf{F}$. Then the paddle will rotate around $\mathbf{n}$ with angular speed

$$\omega = \frac{1}{2}(\operatorname{curl}\mathbf{F}) \cdot \mathbf{n} \quad rad/s$$

2

The proof is a little tricky and (at least in 3D) requires Stokes' Theorem. It is much easier to consider a…

Duck Race!   Suppose that the water in a river of width 4 has velocity $\mathbf{v} = \left(1 - \frac{1}{4}y^2\right)\mathbf{i}$. The curl of the flow is

$$\nabla \times \mathbf{v} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} 1 - \frac{1}{4}y^2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}y \end{pmatrix}$$

Release several ducks and assume that they move with the water.

There are two aspects to the motion of the ducks:

1. Linear. The ducks follow the direction of flow to the right. The faster ducks are closer to midstream.

2. Rotational. The side of a duck closer to midstream will move faster than the side nearer the edge of the river. This will cause most ducks to rotate. According to the Theorem, and interpreting the right hand rule, a duck with $y = 1$ will rotate with angular velocity $\frac{1}{2}\nabla \times \mathbf{v} = \frac{1}{4}\mathbf{k}$, i.e. counter-clockwise at 0.25 radians per second.

Combine the two and you get the motion in the animation.[2] If you've ever been white-water rafting this explains why the guide always wants to stay in the center of the river: you travel fastest and you don't spin!

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[2]Click it to make it move after opening the pdf in Acrobat on your desktop.

Global Rotation ≠ Local Rotation   It is important to distinguish between local rotation (changing the direction an object faces) and global rotation (vector fields making objects travel in loops). Compare the following three duck races. In all cases the ducks will travel counter-clockwise around the origin (global rotation). Their local rotation changes each time.

$$\mathbf{F}_1 = \frac{1}{(x^2 + y^2)^{1/2}} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

$$\nabla \times \mathbf{F}_1 = \frac{1}{(x^2 + y^2)^{1/2}} \mathbf{k}$$

Ducks have local rotation counter-clockwise.
Because of the global rotation of the vector field, there are more arrows on the side of the duck further from the origin. Since all arrows have the same length, the duck rotates counter-clockwise.

$$\mathbf{F}_2 = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

$$\nabla \times \mathbf{F}_2 = \mathbf{0}$$

Ducks have zero local rotation.
The fact that the speed of flow is faster nearer the origin exactly balances the the fact that more arrows are on the outside of the duck.

$$\mathbf{F}_3 = \frac{1}{(x^2 + y^2)^{3/2}} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

$$\nabla \times \mathbf{F}_3 = \frac{-1}{(x^2 + y^2)^{3/2}} \mathbf{k}$$

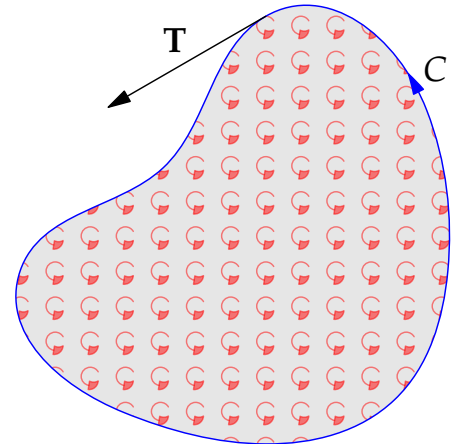Ducks have local rotation clockwise.
Now the speed of flow is so much greater nearer the origin that the duck spins the opposite way.

4

Circulation: Green's Theorem Revisited

Suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field in two dimensions. Then $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, forms part of Green's Theorem, which we can now rewrite:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$
$$= \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

Recall that $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds$ where $\mathbf{T}$ is the unit tangent vector of C. This is often called the *circulation* of the vector field $\mathbf{F}$ around C: how vigorously $\mathbf{F}$ pushes a particle round C.

Interpreting curl as causing local rotation, the double integral $\iint_D \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$ may be interpreted as the sum of all the infinitessimal rotations *inside C*.

To summarize, Green's Theorem says: sum of infinitessimal rotations = total circulation.

Divergence

Definition. *If* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ *is a three-dimensional vector field then the* divergence *of* $\mathbf{F}$ *is the* scalar field

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

defined wherever all partial derivatives exist.[3]

Notation   The dot product notation is completely natural when ∇ is viewed as a differential operator

$$\nabla \cdot \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Example

$$\nabla \cdot \mathbf{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{pmatrix} x^2 - 3y \\ xz \\ x + yz \end{pmatrix} = \frac{\partial}{\partial x}(x^2 - 3y) + \frac{\partial}{\partial y}xz + \frac{\partial}{\partial z}(x + yz) = 2x + y$$

The following easy result is analogous to the fact that $\nabla \times \nabla f = 0$.

Theorem. *If* $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ *has continuous second-derivatives, then* $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$ *(often written* $\nabla \cdot \nabla \times \mathbf{F} = 0$*)*

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[3]Similarly $\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ for a two-dimensional vector field.

Proof.

$$\operatorname{div}\operatorname{curl}\mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix} = \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} = 0 \qquad \blacksquare$$

Laplacians

We have $\nabla \times \nabla f = 0$ and $\nabla \cdot \nabla \times \mathbf{F} = 0$. Only two other combinations make sense:[4] $\nabla(\nabla \cdot \mathbf{F})$ and $\nabla \cdot \nabla f$. The first is not commonly used, but the latter is very important in Physics.

Definition. *If f is a scalar function then its* Laplacian *is the expression*

$$\nabla^2 f = \nabla \cdot \nabla f = \operatorname{div}\operatorname{grad} f = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 R}{\partial z^2}$$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ *we define*

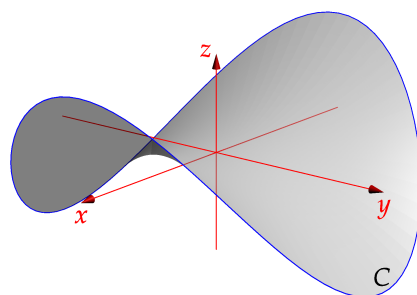$$\nabla^2 \mathbf{F} = \nabla^2 P\mathbf{i} + \nabla^2 Q\mathbf{j} + \nabla^2 R\mathbf{k}$$

Example $\nabla^2(3x^2y^3z + e^{-x^2-y^2}) = 6y^3z + 18x^2yz + 4(x^2 + y^2 - 1)e^{-x^2-y^2}$

Physics applications Among the many places where Laplacians appear in real life are the following.

1. *Minimal surfaces*: Suppose that a wire in the shape of a curve C, parameterized by $\mathbf{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ \cos 2t \end{pmatrix}$ for $0 \le t \le 2\pi$ is dipped in soap. In the absence of gravity, the soap film formed will be the saddle-surface with equation $z = y^2 - x^2$. This satisfies *Laplace's equation*

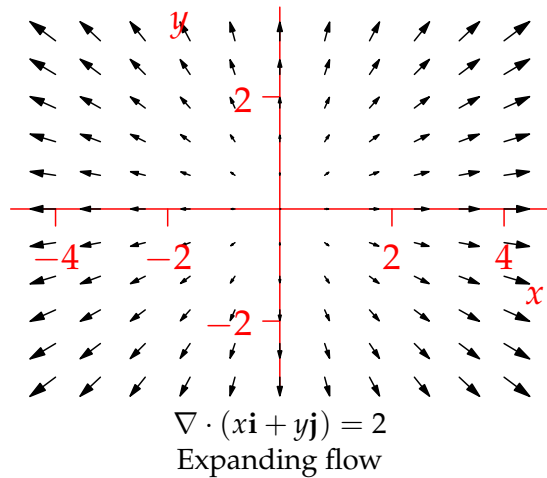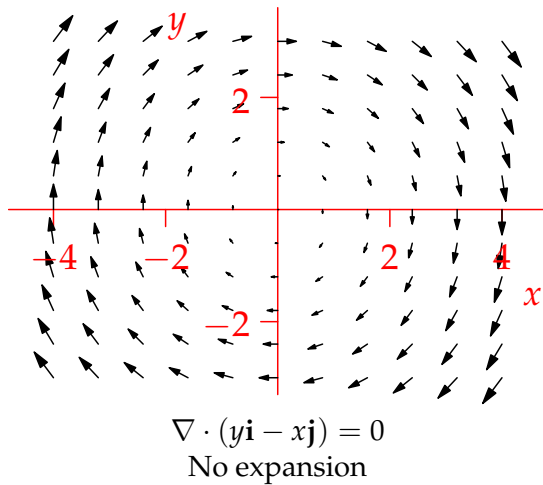$$\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

   A surface formed by a soap bubble has minimal area out of all surfaces with boundary C.

2. *The Heat Equation*: The temperature $T(x, y, z; t)$ at position (x, y, z) and time t satisfies the partial differential equation $\frac{\partial T}{\partial t} = k\nabla^2 T$ $(k > 0$ constant).

3. *The Wave Equation*: The height $u(x, y; t)$ of a wave at position (x, y) and time t satisfies the PDE $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$ $(c$ constant), where c is the *speed* of the wave.

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[4]Think about what is a vector field and what is a scalar…

Expanding Fields: Interpretation of Divergence   Divergence measures the tendency of a vector field to make things *expand.*
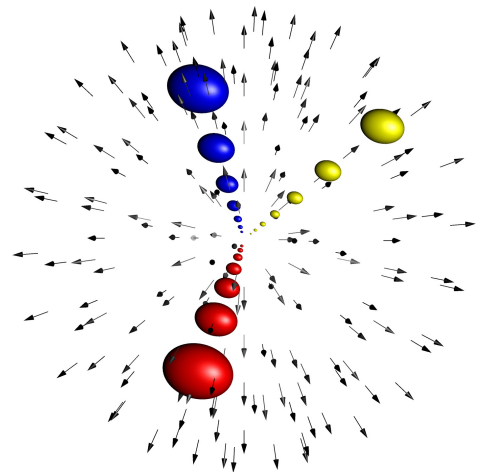

$$\nabla \cdot (y\mathbf{i} - x\mathbf{j}) = 0$$
No expansion

$$\nabla \cdot (x\mathbf{i} + y\mathbf{j}) = 2$$
Expanding flow

Definition.   *A vector field* $\mathbf{F}$ *is* incompressible *if* div $\mathbf{F} = 0$ *everywhere.*

Local expansion   Again the picture is too simplistic: consider *local* expansion which, in three-dimensions, means increase in *volume.*

Theorem.   *Place an infinitessimally balloon in a vector field* $\mathbf{F}$. *Assume that the balloon drifts with the vector field along a path parameterized by* $\mathbf{r}(t)$ *and that the surface of the balloon moves with the vector field. Then the balloon's volume* $V(t)$ *satisfies the differential equation*

$$\frac{1}{V(t)} \frac{\mathrm{d}V}{\mathrm{d}t} = \operatorname{div} \mathbf{F}(\mathbf{r}(t))$$

The larger the divergence, the greater the rate of volume increase. For a two-dimensional vector field, replace the balloon with a tiny oil drop and volume with area. A proof will have to wait until we have the divergence theorem.

Example   The vector field $\mathbf{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{r}$ has constant divergence 3. It follows that an infinitessimal balloon released into the field will see its volume satisfy $V'(t) = 3V(t)$. This is the natural growth equation, with solution

$$V(t) = V_0 e^{3t}$$

where V_0 is the initial volume of the balloon.

Global Expansion $\neq$ Local Expansion   Similarly to our discussion of local rotation and curl, we distinguish between global expansion (particles seeming to travel away from something) and local expansion (increase in volume). In the following animations, pretend that the duck is an oil-drop moving with the field. Divergence effects the volume of the duck (seen here as area).

$$\mathbf{F}_1 = \frac{-1}{r} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \qquad r = \sqrt{x^2 + y^2}$$

$$\nabla \cdot \mathbf{F}_1 = \frac{-1}{r} < 0$$

Duck/oil-drop shrinks as it moves towards origin.
The arrows bring all points in towards the center and reduce volume.

$$\mathbf{F}_2 = \frac{-1}{r^2} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \qquad \nabla \cdot \mathbf{F}_2 = 0$$

Duck's volume unchanged.
The vector field gets stronger nearer the origin. The duck stetches out, perfectly balancing the squashing of the vector field as it moves nearer the origin.

$$\mathbf{F}_3 = \frac{-1}{r^3} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \qquad \nabla \cdot \mathbf{F}_3 = \frac{1}{r^3} > 0$$

Duck's volume increases!
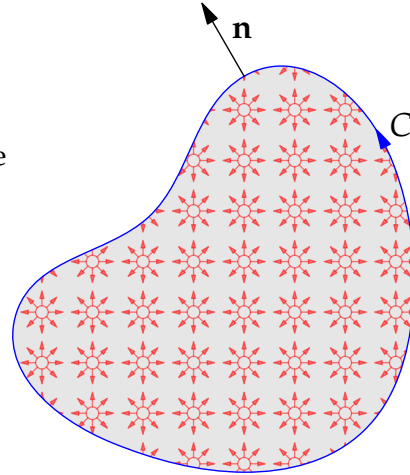The increase in speed is so great near the origin, that the duck gets longer faster than it narrows.

Green's Theorem, version III We can also interpret Green's Theorem using divergence of a 2-dimensional vector field.

Suppose that $\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $a \le t \le b$ parameterizes a positively-oriented, simple, closed curve. It's *outward pointing normal vector* $\mathbf{n}$ must be the unit tangent vector $\mathbf{T} = \frac{1}{|\mathbf{r}'|}\mathbf{r}'$ rotated $90°$ clockwise. That is,

$$\mathbf{n}(t) = \frac{1}{|\mathbf{r}'(t)|}\begin{pmatrix} y'(t) \\ -x'(t) \end{pmatrix}$$

We now compute a new line integral, and convert to a double integral using Green's Theorem.

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \int_a^b \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \frac{1}{|\mathbf{r}'|}\begin{pmatrix} y' \\ -x' \end{pmatrix}|\mathbf{r}'|\,dt$$
$$= \oint_C -Q\,dx + P\,dy$$
$$= \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\,dA = \iint_D \operatorname{div}\mathbf{F}\,dA$$

$\mathbf{F} \cdot \mathbf{n}$ measures how much vector field $\mathbf{F}$ points across the boundary C. The integral $\oint_C \mathbf{F} \cdot \mathbf{n}\,ds$ therefore measures the total flow rate of $\mathbf{F}$ out of C. This is called the *flux* of the vector field $\mathbf{F}$ out of C. Since divergence measures the tendency of a vector field to cause expansion, the integral $\iint_D \operatorname{div}\mathbf{F}\,dA$ is the sum of all the infinitessimal expansions *inside C*.

Green's Theorem can therefore be interpreted: sum of infinitessimal expansions = total outward flow.[5]

Summary

Nabla $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ is a differential operator

Gradient $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$ is a *vector field*
    Measures direction and magnitude of greatest increase in f

Curl $\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$ is a *vector field*
    Measures local rotation of $\mathbf{F}$ in 3D
    $\frac{1}{2}\operatorname{curl}\mathbf{F} \cdot \mathbf{n}$ is the angular speed of flow around axis $\mathbf{n}$

Divergence $\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ is a *scalar field*
    Measures local rate of change of volume
    $\operatorname{div}\mathbf{F} > 0 \implies$ expansion, $\operatorname{div}\mathbf{F} < 0 \implies$ contraction

Identities $\nabla \times \nabla f = \mathbf{0}$, $\nabla \cdot \nabla \times \mathbf{F} = 0$

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[5]This is precisely the Divergence Theorem in 2-dimensions.